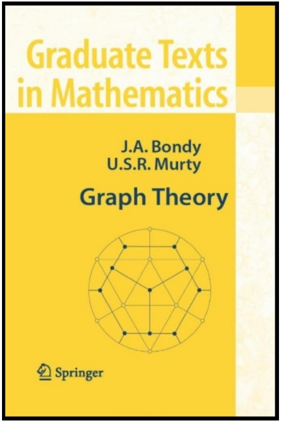
Graph Theory

Chapter 6. Tree-Search Algorithms

6.1. Tree-Search—Proofs of Theorems



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(a) for every vertex v of G, $\ell(v) = d_T(r, v)$, the level of v in T, and

(b) every edge of G joins vertices on the same or consecutive levels of T; that is, $|\ell(u) - \ell(v)| \leq 1$ for all $uv \in E$.

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Proof. (a) This is to be proved (inductively) in Exercise 6.1.1.

(b) If uv is and edge of G joining u and v on the same level, then by the definition of "level," $\ell(u) = \ell(v)$ as claimed. So it suffices to prove that if $uv \in E$ and $\ell(u) < \ell(v)$ then $\ell(u(= \ell(v) - 1$.

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First, we prove by induction on $\ell(u)$ that if u and v satisfy $\ell(u) < \ell(v)$, then u is joined to A "before" (in terms of the values of $t(u)$ and $t(v)$). For the base case, if $\ell(u) = 0$ then $u = r$ is the root of the tree, $t(u) = 1$, and for every other vertex $i > 1$ (by Step 9) and so every other vertex v satisfies $t(v) > 1$ (by Step 11).

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Theorem 6.2 (continued 1)

Proof (continued). For the induction hypothesis, suppose that the assertion is true whenever $\ell(u) < k$, and consider the case $\ell(u) = k$ where $k > 0$. Notice that $0 < k = \ell(u) < \ell(v)$ so that neither u nor v is the root. Use the predecessor function p to define $x = p(u)$ and $y = p(v)$. Since predecessors in a rooted tree are unique, then $x \neq y$. By Step 11 we have $\ell(x) = \ell(u) - 1$ (with y of Step 11 equal to u here) and $\ell(y) = \ell(v) - 1$ (with x and y of Step 11 equal to y and v, respectively, here). Since we consider $\ell(u) < \ell(v)$, then we have $\ell(x) = \ell(u) - 1 < \ell(v) - 1 - \ell(y)$. Since $\ell(u) = k$ then $\ell(x) = k - 1 < k$ and so by the induction hypothesis we have that x joined Q before y (i.e., $t(x) < t(y)$). Since u is a neighbor of x in T (by the predecessor function) and $u \neq v$ then u joined Q before v (i.e., $t(u) < t(v)$). Therefore, by mathematical induction, if $\ell(u) < \ell(v)$ then u joined Q before v.

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Proof (continued). Now consider $uv \in E$ with $\ell(u) < \ell(v)$. (i) If $u = p(v)$, then by Step 11 we have $\ell(u) = \ell(v) - 1$ (with x and y of Step 11 equal to u and v, respectively, here), and the claim follows. (ii) If $u \neq p(v)$ then set $y = p(v)$. Then v was added to T by the edge yv (by Steps 11 and 12) and not by the edge uv. So vertex y joined Q before u, for if u had joined Q before y (and hence before v), then the fact that v is a neighbor of u means that v would have joined Q before y by Step 8 (with v as an uncoloured neighbor of u) and Step 12 (with y there equal to v here), contradicting the fact that y is a predecessor of v. So by the first part of the proof of (b), $\ell(y) \leq \ell(u)$.

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Theorem 6.3

Theorem 6.3. Let G be a connected graph. Then the values of the level function ℓ returned by BFS are the distances in G from the root r: $\ell(v) = d_G(r, v)$ for all $v \in V$.

Proof. By Theorem 6.2(a), $\ell(v) = d_T(r, v)$. Now $d_T(r, v) \geq d_G(r, v)$ because T is a subgraph of G. Thus $\ell(v) \geq d_G(r, v)$. We now reverse this inequality by an induction argument on the length of the shortest (r, v)-path in G.

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Let P be a shortest (r, v)-path in G, where $r \neq v$. If P is of length 1 then v is adjacent to r in G. By Steps 3 through 12, the root r is first added to Q and then all neighbors of r are added to T at level 1. This establishes the base case for the induction argument. Now, for the induction hypothesis, suppose the inequality $\ell(v) \leq d_G(r, v)$ holds for shortest (r, v)-path P in G of length k. Consider vertex v in G such that the shortest (r, v)-path P in G is of length $k + 1$. Let u be the predecessor of v on P (not to be confused with the predecessor $p(v)$ in T).

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Proof (continued). Then rPu is a shortest (r, u)-path in G, and $d_G(r, u) = d_G(r, v) = 1 = k$. So by the induction hypothesis, $\ell(u) \leq d_G(r, u)$. Since u and v are adjacent in G, then by Theorem 6.2(b) we have $\ell(v) - \ell(u) \leq 1$. Therefore $\ell(v) \leq \ell(u) + 1 \leq d_G(r, u) + 1 = d_G(r, v)$.

So the induction step holds and by mathematical induction $\ell(v) \leq d_G(r, v)$ for all vertices v in G. Combining this with the first inequality gives the desired equality. □

Proposition 6.5

Proposition 6.5. Let u and v be two vertices of G, with $f(u) < f(v)$.

(a) If u and v are adjacent in G, then $l(u) < l(v)$.

(b) u is an ancestor of v in T if and only if $l(v) < l(u)$.

Proof. (a) Informally, this claim is that if u is added to the top of stack S before its neighbor v (i.e., $f(u) < f(v)$), then v leaves the stack before u (i.e., $l(u) < l(v)$). By Steps 12 and 13, u is added to stack S at time $f(u)$. By Steps 10–13, all uncolored neighbors are considered for addition to S before vertex u is removed from S. Parameter i is incremented by 1 in Step 9 after each uncolored neighbor of u is considered for addition to S.

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Proof (continued). (b) First, suppose that u is an ancestor of v in T. By Steps 9 and 12, the values of f increase along the path uTv (i is incremented in Step 9 and f values are assigned to vertices using the value of i and the predecessor function in Step 12). That is, $f(u) < f(v)$. So by part (a) to each vertex in path uTv we have $l(u_< l(v)$ (we have to apply to consecutive vertices in the path since part (a) requires that we compare neighbors), as claimed.

Conversely, suppose that u is not an ancestor of v in T. Since $f(u) < f(v)$ by hypothesis then v is not an ancestor of u wither, it could lie on another "branch" of the family tree). So u does not lie on the path rTv and v does not lie on the path rTu. Let s be the last common vertex of these two paths.

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Proof (continued). Since $f(u) < f(v)$ then u was added to stack S before v, and hence all the proper descendants of s on path rTv can be added to S only after all the proper descendants of s on the path rTu have been added and removed from S (after which s is the top vertex). In particular, v can only be added to S (at time $f(v)$) after u has been removed (at time $l(u)$), so that $l(u) < f(v)$. Since v is added to S before it is removed then $f(v) < l(v)$. Therefore $l(u) < f(v) < l(v)$. So if u is not an ancestor of v in T then $l(u) < l(v)$. Since $u \neq v$ then $l(u) \neq l(v)$, so the contrapositive of what we have shown is that if $l(u) \geq l(v)$ (that is, if $l(v) < l(u)$) then u is an ancestor of v in T, as claimed. □

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Proposition 6.6

Proposition 6.6. Let T be a DFS-tree of a graph G. Then every edge of G joins vertices which are related in T.

Proof. Let uv be any edge of G. Without loss of generality, say $f(u) < f(v)$. By Proposition 6.5(a) we have $l(v) < l(u)$. By Proposition 6.5(b), u is an ancestor of v and so (by definition) u and v are related in T. □

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