Graph Theory

Chapter 6. Tree-Search Algorithms

6.2. Minimum-Weight Spanning Trees—Proofs of Theorems

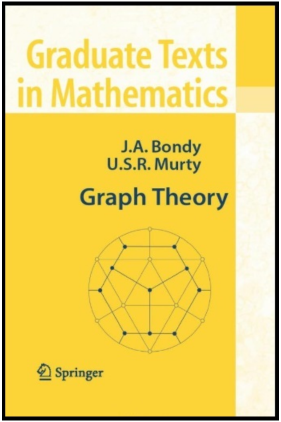

Table of contents

Theorem 6.10

Theorem 6.10. Every Jarník-Prim tree is an optimal tree.

Proof. Let T be a Jarník-Prim tre. We give an inductive proof on $v(T)$, the number of vertices in an optimal (spanning) tree. If $v(T) = 1$ or $v(T) = 2$, then the result holds trivially, giving the base case. For the induction hypothesis, suppose all Jarník-Prim trees on k vertices are optimal trees and let T be a Jarník-Prim tree on $v(T) = k + 1$ vertices.

Theorem 6.10

Theorem 6.10. Every Jarník-Prim tree is an optimal tree.

Proof. Let T be a Jarník-Prim tre. We give an inductive proof on $v(T)$, the number of vertices in an optimal (spanning) tree. If $v(T) = 1$ or $v(T) = 2$, then the result holds trivially, giving the base case. For the induction hypothesis, suppose all Jarník-Prim trees on k vertices are optimal trees and let T be a Jarník-Prim tree on $v(T) = k + 1$ vertices.

The first edge added to T is an edge e of least-weight in the edge cut associated with $\{r\}$ by Step 5. That is, $w(e) \leq w(f)$ for all edges f incident with r. We first claim that some optimal tree includes this edge e. Let T^* be an optimal tree. If $e \in E(T^*)$ then our claim holds, so suppose $e \notin E(T^*)$. Thus $T^* + e$ contains a cycle C (this follows from Proposition 4.1 since in T^* there is a unique path connecting the ends of e).

Theorem 6.10

Theorem 6.10. Every Jarník-Prim tree is an optimal tree.

Proof. Let T be a Jarník-Prim tre. We give an inductive proof on $v(T)$, the number of vertices in an optimal (spanning) tree. If $v(T) = 1$ or $v(T) = 2$, then the result holds trivially, giving the base case. For the induction hypothesis, suppose all Jarník-Prim trees on k vertices are optimal trees and let T be a Jarník-Prim tree on $v(T) = k + 1$ vertices.

The first edge added to T is an edge e of least-weight in the edge cut associated with $\{r\}$ by Step 5. That is, $w(e) \leq w(f)$ for all edges f incident with r. We first claim that some optimal tree includes this edge e. Let T^* be an optimal tree. If $e \in E(T^*)$ then our claim holds, so suppose $e \notin E(T^*)$. Thus $T^* + e$ contains a cycle C (this follows from Proposition 4.1 since in T^* there is a unique path connecting the ends of e).

Theorem 6.10 (continued 1)

Theorem 6.10. Every Jarník-Prim tree is an optimal tree.

Proof (continued). Let f be the other end of C incident with r. Then $T^{**} = (T^* + e) \setminus f$ is a spanning tree of G. Since $w(e) \leq w(f)$ as described above, then $w(T^{**}) = w(T^*) + w(e) - w(f) \leq w(T^*)$. Since T^* is an optimal tree (that is, a minimum weight spanning tree) then T^{**} is also an optimal tree containing e and our first claim holds.

Now consider the graph $G' = G/e$ (that is, contract edge e in G) and denote by r' the vertex resulting from the contraction of e. By Exercise 4.2.1(a), there is a one-to-one correspondence (i.e., a bijection) between the set of spanning trees of G that contain e and the set of spanning trees of G'. Set $T' = T/e$. Our second claim is that T' is a Jarník-Prim tree of G' rooted at r'.

Theorem 6.10 (continued 1)

Theorem 6.10. Every Jarník-Prim tree is an optimal tree.

Proof (continued). Let f be the other end of C incident with r. Then $T^{**} = (T^* + e) \setminus f$ is a spanning tree of G. Since $w(e) \leq w(f)$ as described above, then $w(T^{**}) = w(T^*) + w(e) - w(f) \leq w(T^*)$. Since T^* is an optimal tree (that is, a minimum weight spanning tree) then T^{**} is also an optimal tree containing e and our first claim holds.

Now consider the graph $G' = G/e$ (that is, contract edge e in G) and denote by r' the vertex resulting from the contraction of e. By Exercise 4.2.1(a), there is a one-to-one correspondence (i.e., a bijection) between the set of spanning trees of G that contain e and the set of spanning trees of G'. Set $T' = T/e$. Our second claim is that T' is a Jarník-Prim tree of G' rooted at r'.

Theorem 6.10 (continued 2)

Theorem 6.10. Every Jarník-Prim tree is an optimal tree.

Proof (continued). At any step in the creation of T as a Jarník-Prim tree with root r (say the tree at this step is T_s; we assume T_s includes edge e by the first claim), let $T_s' = T_s/e$. Then $\partial(T_s) = \partial(T_s')$. so an edge of minimum weight in $\partial(T_s)$ is also an edge of minimum weight in $\partial(T_s')$. In Jarník-Prim tree T the edge added to T_s is of minimum weight in $\partial(T_s)$ (by Step 5), so the same edge may be added to T_s if the Jarník-Prim algorithm is applied to it at this step. Since the final tree T is a Jarník-Prim tree of G with root r, then T' is a Jarník-Prim tree of G with root r', as claimed. Since G' has k vertices, then T' is an optimal tree by the induction hypothesis. Since e is a least-weight edge incident to r in G, then T is an optimal tree of G, and the induction step holds. Therefore, by mathematical induction, the result holds for all connected graphs G. □

Theorem 6.10 (continued 2)

Theorem 6.10. Every Jarník-Prim tree is an optimal tree.

Proof (continued). At any step in the creation of T as a Jarník-Prim tree with root r (say the tree at this step is T_s; we assume T_s includes edge e by the first claim), let $T_s' = T_s/e$. Then $\partial(T_s) = \partial(T_s')$. so an edge of minimum weight in $\partial(T_s)$ is also an edge of minimum weight in $\partial(T_s')$. In Jarník-Prim tree T the edge added to T_s is of minimum weight in $\partial(T_s)$ (by Step 5), so the same edge may be added to T_s if the Jarník-Prim algorithm is applied to it at this step. Since the final tree T is a Jarník-Prim tree of G with root r, then T' is a Jarník-Prim tree of G with root r', as claimed. Since G' has k vertices, then T' is an optimal tree by the induction hypothesis. Since e is a least-weight edge incident to r in G, then T is an optimal tree of G, and the induction step holds. Therefore, by mathematical induction, the result holds for all connected graphs G. □