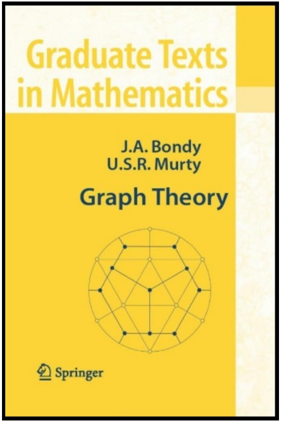
Graph Theory

Chapter 9. Connectivity

9.1. Vertex Connectivity—Proofs of Theorems



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Theorem 9.1. Menger's Theorem

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Proof. The proof is based on induction on the number of edges of G. First, if G has 3 vertices then G is a path of length 2 with x and y as its ends and $p(x, y) = c(x, y) = 1$; this is the base case. Now suppose the result holds for all graphs on less than m edges and let $e(G) = m$.

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First, "for convenience" we denote $k = c(x, y) = c_G(x, y)$. Now any xy-path in G must meet at least one vertex of an xy-cut (or else the xy-cut is not an xy-cut since there would be a path connecting x and y after the deletion of the xy-cut). So in a family $\mathcal{P}$ of internally disjoint xy-paths, the paths meet an xy-cut in at least $|\mathcal{P}|$ vertices. Hence $|\mathcal{P}| \leq k$ and $p(x, y) = p_G(x, y) \leq k = c(x, y)$.

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Theorem 9.1. Menger's Theorem (continued 1)

Proof (continued). So we need to show that $p_G(x,y) \geq k = c(x,y)$. We may assume that there is an edge $e = uv$ incident to neither x nor y (otherwise every xy-path is of length two and then the number of internally disjoint paths equals the size of the xy-cut, since each vertex in the xy-cut is the center of one of the internally disjoint 2-paths and conversely). So $H = G \setminus e$ so that $e(H) = e(G) - 1 = m - 1$.

Because H is a subgraph of G then $p_H(x,y) \leq p_G(x,y)$. By the induction hypothesis, $p_H(x,y) = c_H(x,y)$. Now an xy-vertex-cut of $H = G \setminus e$ along with either end of e is an xy-vertex-cut of G (since the only difference between G and H is the edge e, and when an xy-vertex-cut of H along with an end of e is deleted from G, the result is graph H with the xy-vertex-cut of H deleted). Since $c_H(x,y)$ denotes the minimum size of a vertex cut separating x and y in H then $c_G(x,y)$ is at most $c_H(x,y)+1$; i.e., $c_G(x,y) \leq c_H(x,y)+1$. Therefore

$$p_G(x,y) \geq p_H(x,y) = c_H(x,y) \geq c_G(x,y) - 1 = k - 1. \qquad (*)$$

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Theorem 9.1. Menger's Theorem (continued 2)

Proof (continued). ...

$$p_G(x, y) \geq p_H(x, y) = c_H(x, y) \geq c_G(x, y) - 1 = k - 1. \qquad (*)$$

As discussed above, $p_G(x, y) \leq k$ so if $p_G(x, y) \geq k$ then we have $p_G(x, y) = k = c_G(x, y)$ and we are done. So without loss of generality we can suppose the inequalities in $(*)$ are equalities so that, in particular, $c_H(x, y) = k - 1$. So let $S = \{v_1, v_2, \ldots, v_{k-1}\}$ be a minimum xy-vertex-cut in H. Let X be the set of vertices reachable from x in $H - S$ (so $y \notin X$), and let Y be the set of vertices reachable from y in $H - S$ (so $x \notin Y$). Because $|S| = k - 1 < k$, the set S is not an xy-vertex-cut of G, so there is an xy-path in $G - S$. This path necessarily includes edge e (or else it would be an xy-path in $H - S$ and S would not be an xy-vertex-cut of H). So one end of e is reachable from x in $H - S$ and the other end of e is reachable from y in $H - S$; say $u \in X$ and $v \in Y$.

Theorem 9.1. Menger's Theorem (continued 2)

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Theorem 9.1. Menger's Theorem (continued 3)

Proof (continued).

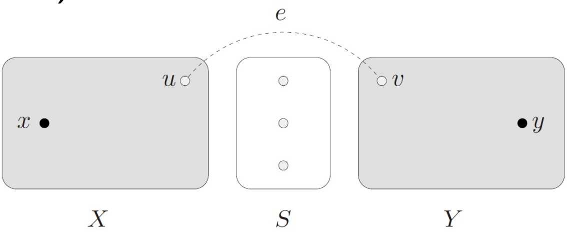


Figure 9.3(a). Sets X and Y, and edge e.

Now consider the graph G/Y obtained from G by shrinking Y to a single vertex y. Every xy-vertex cut T in G/Y is also an xy-vertex cut in G, because if P were an xy-path in G which avoids T then the subgraph P/Y of G/Y would contain an xy-path in G/Y which avoids T contradicting the property of T as an xy-vertex-cut in G/Y. So the minimum size of an xy-vertex-cut in G/Y is at least as big as the minimum size of an xy-vertex-cut in G: $c_{G/Y} \geq c_G(x, y) = k$.

Theorem 9.1. Menger's Theorem (continued 4)

Proof (continued). On the other hand, $c_{G/Y}(x, y) \leq k$ because $S \cup \{u\}$ (where $|S \cup \{u\}| = k$) is an xy-vertex-cut of G/Y:

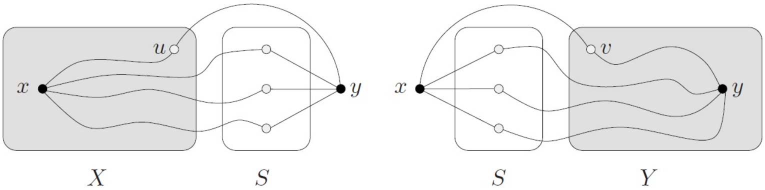


Figure 9.3(b). Graphs G/Y (left) and G/X (right).

So $S \cup \{u\}$ is a minimum xy-vertex-cut of G/Y and $c_{G/Y}(x, y) = k$. To conclude the proof, we will find k internally disjoint xy-paths in G/Y and from these produce k internally disjoint xy-paths in G.

Theorem 9.1. Menger's Theorem (continued 5)

Proof (continued). Now vertex y of G is reachable from v (see Figure 9.1(a)) and $v \neq y$, so the number of edges of G/Y is less than the number of edges in G. So by the induction hypothesis, there are $k = p_{G/Y}(x, y)$ internally disjoint xy-paths. Now the neighbors of y in G/Y are $v_1, v_2, \ldots, v_{k-1}, u$. So the k internally disjoint xy-paths in G/Y, $P_1, P_2, \ldots, P_k$, must have the property that each vertex of $S \cup \{u\}$ lies on one of them (see Figure 9.1(b) left). We take $v_i \in V(P_i)$ for $1 \leq i \leq k-1$ and $u \in P_k$.

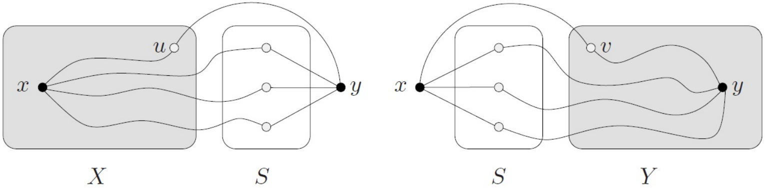
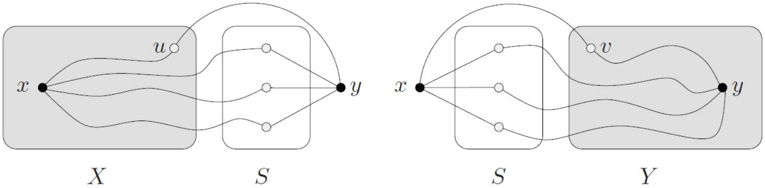


Figure 9.3(b). Graphs G/Y (left) and G/X (right).

Theorem 9.1. Menger's Theorem (continued 6)

Proof (continued). Likewise, there are k internally disjoint xy-paths $Q_1, Q_2, \ldots, Q_k$ in G/X obtained by shrinking X to x with $v_i \in V(Q_i)$ for $1 \leq i \leq k-1$ and $v \in Q_k$ (see Figure 9.1(b) right).



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We then have k internally disjoint xy-paths in G, namely $xP_iv_iQ_iy$ for $1 \leq i \leq k-1$, and xP_kuvQ_ky (see Figure 9.3(c)).

Theorem 9.1. Menger's Theorem (continued 7)

Proof (continued). We then have k internally disjoint xy-paths in G, namely $xP_iv_iQ_iy$ for $1 \le i \le k-1$, and xP_kuvQ_ky (see Figure 9.3(c)). So $p_G(x,y) = c_G(x,y) = k$ and the result holds for graphs with m edges. Therefore, by mathematical induction on the number of edges of a graph, the claim holds for all graphs. □

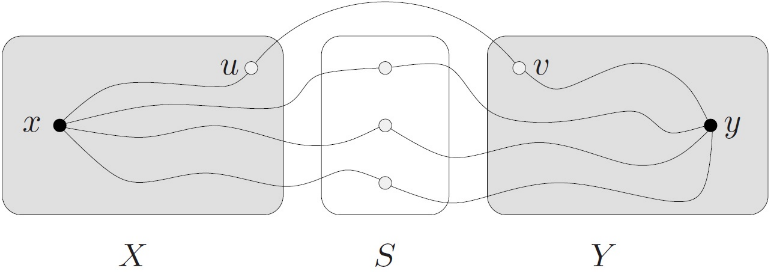

Figure 9.3(c). k internally disjoint xy-paths in G.

Theorem 9.2

Theorem 9.2. If G has at least one pair of nonadjacent vertices, then

$$\kappa(G) = \min\{p(u,v) \mid u, v \in V, u \neq v, uv \notin E\}. \tag{9.3}$$

Proof. Notice that loops do not affect $p(u,v)$. As observed in Note 9.1.B, $p(u,v)$ is not affected by parallel edges when u and v are not adjacent. Hence, we may assume without loss of generality that G is simple.

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Consider the graph $H = G \setminus xy$ obtained by deleting edge xy from G. Since G is simple, then $p_G(x,y) = p_H(x,y) + 1$. By Menger's Theorem (Theorem 9.1), $p_H(x,y) = c_H(x,y)$. Let X be a minimum vertex cut in H separating x and y so that $p_H(x,y) = c_H(x,y) = |X|$. Hence $p_G(x,y) = |X| + 1$.

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Theorem 9.2 (continued 1)

Proof (continued). ASSUME $V \setminus X = \{x, y\}$. Then

$$\begin{aligned} \kappa(G) &= p_G(x,y) \text{ by the choice of } x \text{ and } y \\ &= |X| + 1 \\ &= (n-2) + 1 \text{ since } V \setminus X = \{x, y\} \\ &= n - 1. \end{aligned}$$

But if $\kappa(G) = n - 1$ then there are $n - 1$ internally disjoint paths from x to y; these include the edge xy and all paths of length 2 from x to y and through a third vertex. Since $n - 1$ is a minimum for $p_G(x, y)$ then G must be complete, but this CONTRADICTS the hypothesis that G has a pair of nonadjacent vertices. So the assumption that $V \setminus X = \{x, y\}$ is false and hence there must be a vertex z of G such that $\{x, y, z\} \subseteq V \setminus X$.

Theorem 9.2 (continued 2)

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