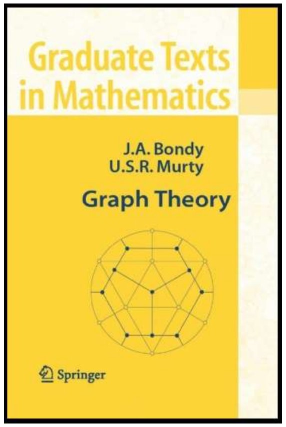
Graph Theory

Chapter 9. Connectivity

9.2. The Fan Lemma—Proofs of Theorems



Lemma 9.3

Lemma 9.3. Let G be a k-connected graph and let H be a graph obtained from G by adding a new vertex y and joining it to at least k vertices of G. Then H is also k-connected.

Proof. In any two vertices of graph H are adjacent (in which case the underlying simple graph of H is a complete graph), then the connectivity of H is one less than the number of vertices of H (by Note 9.1.A). Since $v(H) \geq k - 1$, then $\kappa(H) \geq k$ and so H is k-connected. Now suppose $S \subset V(H)$ with $|S| = k - 1$. We'll show that $H - S$ is connected and, since S is an arbitrary subset of $V(H)$, then $\kappa(H) \geq k$ and so H is k-connected.

First, suppose $y \in S$. Then $H - S = G - (S \setminus \{y\})$. Since G is k-connected by hypothesis (and so G is j-connected for all $j \leq k$) and $|S \setminus \{y\}| = k - 2$, then $H - S$ is connected, as needed.

Lemma 9.3 (continued)

Lemma 9.3. Let G be a k-connected graph and let H be a graph obtained from G by adding a new vertex y and joining it to at least k vertices of G. Then H is also k-connected.

Proof (continued). Second, suppose $y \notin S$. Now y, by hypothesis, has at least k neighbors in $V(G)$. Since $|S| = k - 1$, there is a neighbor z of y that is not in S. Since G is (by hypothesis, k-connected then $G - S$ is connected and so $(G - S) + yz$ is connected. Since z and y are vertices of $G - S$ and z is a neighbor of y, then yz is an edge of $H - S$. So $(G - S) + yz$ is a connected spanning subgraph of $H - S$. Hence $H - S$ is connected, as needed. □

Proposition 9.4

Proposition 9.4. Let G be a k-connected graph, and let X and Y be subsets of V of cardinality at least k. Then there exists in G a family of k pairwise disjoint (X, Y)-paths.

Proof. Create graph H by adding vertices x and y to graph G and joining x to each vertex of X and y to each vertex of Y. By Lemma 9.3 (applied twice), H is k-connected. By Menger's Theorem (Theorem 9.1) there exist k internally disjoint xy-paths in H. Deleting x and y from each of these paths, we obtain k disjoint paths $Q_1, Q_2, \ldots, Q_k$ in G, each of which has its initial vertex in X (since x is adjacent to all elements of X) and the terminal vertex in Y (since y is adjacent to all elements of Y).

Proposition 9.4 (continued)

Proposition 9.4. Let G be a k-connected graph, and let X and Y be subsets of V of cardinality at least k. Then there exists in G a family of k pairwise disjoint (X, Y)-paths.

Proof (continued). Now the Q_i are not necessarily (X, Y)-paths ((X, Y)-paths have internal vertices in neither X nor Y). But every Q_i contains a segment P_i with initial vertex in X, terminal vertex in Y, and no internal vertices in $X \cup Y$ (take P_i as the segment of Q_i from the "last" vertex of P_i in X to the "first" vertex following this one which is in Y). That is, P_i is an (X, Y)-path for $1 \leq i \leq k$, and since the Q_i are internally disjoint then the P_i are pairwise disjoint, as claimed. □

Theorem 9.6

Theorem 9.6. Let X be a set of k vertices in a k-connected graph G, where $k \geq 2$. Then there is a cycle in G which includes all vertices of S.

Proof. We give a proof based on induction on k. Note 9.2.A establishes the base case $k = 2$. So suppose $k \geq 3$ and that the result holds for $k - 1$.

Let $x \in S$, and let $T = S \setminus x$. Since G is k-connected, then G is also $k - 1$ connected. Therefore by the induction hypothesis, there is a cycle C in G which includes the vertices in set T. Set $Y = V(C)$. If $x \in Y$, then C includes all vertices of S, as desired. So without loss of generality we may assume that $x \notin Y$.

If $|Y| \geq k$ then by The Fan Lemma there is a k-fan in G from x to Y. Because $|T| = k - 1$, the set T divides C into $k - 1$ edge-disjoint segments:

Theorem 9.6 (continued 1)

Proof (continued).

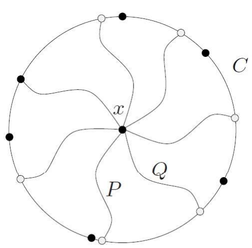


Figure 9.5(a). The k-fan in G is from vertex x to the vertices represented with open circles. The vertices of S and T are represented by "solid" circles.

By the Pigeonhole Principle, some two paths of the fan, P and Q, end in the same one of these segments. The subgraph $C \cup P \cup Q$ contains three cycles, one of which includes $S = T \cup \{x\}$ (see Figure 9.5(b)).

Theorem 9.6 (continued 2)

Proof (continued).

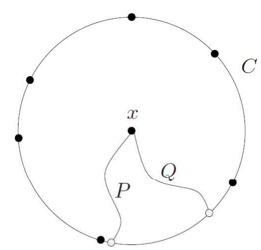


Figure 9.5(b). The vertices of S are solid and are all in the same cycle.

So the result holds if $|Y| \geq k$.

If $Y = V(C)$ satisfies $|Y| = k - 1$ (which is the minimum $|Y|$ can be since cycle C includes all $k - 1$ vertices of $T = S \setminus x$). Again by The Fan Lemma, there is a $(k - 1)$-fan from x to Y in which each vertex of Y is the terminus of one path (so we have a case line Figure 9.5(a), but all vertices on C are represented by "solid" circles).

Theorem 9.6 (continued 3)

Theorem 9.6. Let X be a set of k vertices in a k-connected graph G, where $k \geq 2$. Then there is a cycle in G which includes all vertices of S.

Proof (continued). As above, we can take P and Q as any "consecutive" edges in the fan and produce a cycle as is done in Figure 9.5(b). So the result holds if $|Y| = k - 1$.

So assuming the result holds for $k - 1$ implies the result holds for k itself. Therefore, by mathematical induction on k, the result holds for all graphs satisfying the hypotheses. □