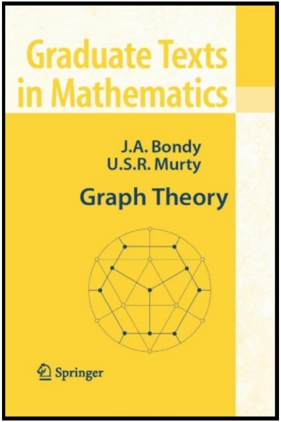
Graph Theory

Chapter 9. Connectivity

9.4. Three-Connected Graphs—Proofs of Theorems



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Theorem 9.9

Theorem 9.9. Let G be a 2-connected graph and let S be a 2-vertex cut of G. Then the marked S-components of G are also 2-connected.

Proof. Let H be a marked S-component of G, with vertex set $S \cup X$ where X is the set of vertices of H that are not in S (notice $|X| \geq 1$ since H actually is a component). Then $|V(H)| = |S| + |X| \geq 3$. Thus if H is complete, it is 2-connected.

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Lemma 9.11

Lemma 9.11. Let G be a 3-connected graph on at least five vertices, and let $e = xy$ be an edge of G such that G/e is not 3-connected. Then there exists a vertex z such that $\{x, y, z\}$ is a 3-vertex cut of G.

Proof. Let $\{z, w\}$ be a 2-vertex cut of G/e (which exists since G/e is hypothesized to not be 3-connected; it has connectivity at most 2). At least one of these two vertices, say z, is not the vertex resulting from the contraction of e. Let $F = G - z$.

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Proof (continued). It now follows from Exercise 9.1.5 that w must be the vertex resulting from the contraction of edge e. Since $e = xy$, then $(G/e) - w = G - \{x, y\}$. Therefore

$$G - \{x, y, z\} = (G - \{x, y\}) - z = (G/e - w) - z = (G/e) - \{z, w\}$$

is disconnected, since we started with $\{z, w\}$ as a 2-vertex cut of G/e. That is, $\{x, y, z\}$ is a 3-vertex cut in G, as claimed. □

Theorem 9.10

Theorem 9.10. Let G be a 3-connected graph on at least five vertices. Then G contains an edge e such that G/e is 3-connected.

Proof. Let G be a 3-connected graph on at least five vertices. ASSUME there is no edge e of G such that G/e is 3-connected. That is, assume for any edge $e = xy$ of G, the contraction G/e is not 3-connected. By Lemma 9.11, there exists a vertex z such that $\{x, y, z\}$ is a 3-vertex cut of G. Then $G - \{x, y, z\}$ has at least two connected components, so choose edge e and vertex z in such a way that $G - \{x, y, z\}$ has a component F with as many vertices as possible.

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Theorem 9.10 (continued 1)

Proof (continued).

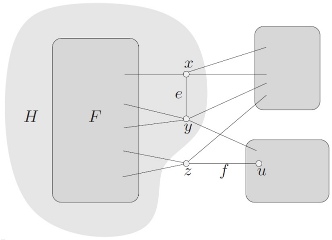

Figure 9.9.

So by Theorem 9.9, the marked $\{x, y\}$-component $H = G[V(F) \cup \{x, y\}]$ is 2-connected.

Let u be a neighbor of z in some component of $G - \{x, y, z\}$ different from F. Since $f = zu$ is an edge of G so, by our assumption, G/f is not 3-connected. By Lemma 9.11 there is a vertex v such that $\{z, u, v\}$ is a 3-vertex cut of G.

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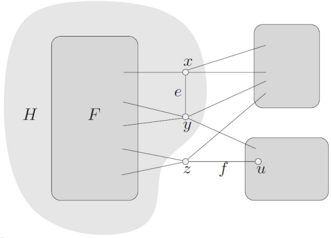

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Theorem 9.10. Let G be a 3-connected graph on at least five vertices. Then G contains an edge e such that G/e is 3-connected.

Proof (continued). Moreover, because H is 2-connected then $H - v$ is connected (similar to the argument above that G being 3-connected implies that $G - z$ is 2-connected); it may be that $v \notin V(H)$ in which case we have $H - v$ as H itself. Since $H - v$ is connected then it is contained in some connected component of $G - \{z, u, v\}$. But then this component has more vertices than F (because H has two more vertices than F, so that $H - v$ has one or two more vertices than F; see Figure 9.9 again). But this CONTRADICTS the choice of edge $e = xy$ and vertex z as yielding F as a component of $G - \{x, y, z\}$ with as many vertices as possible. So the assumption that for every edge e of G, G/e is not 3-connected is false. That is, there is some edge e of G such that G/e is 3-connected, as claimed. □

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Theorem 9.12

Theorem 9.12. Let G be a 3-connected graph, let v be a vertex of G of degree at least four, and let H be an expansion of G at v. Then H is 3-connected.

Proof. Since G is 3-connected then $G - v$ is 2-connected as described in the proofs of both Lemma 9.11 and Theorem 9.10. So by Lemma 9.3 of Section 9.2 (since v_1 and v_2 have at least two neighbors in $G - v$) the graph $H \setminus e$ is 2-connected.

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Let x and y be two vertices of H. If x and y are in $G - v$ then there are three internally disjoint paths in H joining x and y since G is 3-connected (though if one of the paths contains vertex v then we must split vertex v into vertices v_1 and v_2 in that path). If $x \in \{v_1, v_2\}$, say $x = v_1$, and $y \in G - v$ then there are three internally disjoint paths in G joining v and y since G is 3-connected.

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Theorem 9.12 (continued 1)

Proof (continued). Then these three internally disjoint xy-paths in G determine three internally disjoint v_1y-paths in H, where we replace x either with v_1 or with v_1ev_2 as needed (depending on whether the neighbor of x in an xy-path is a neighbor of v_1 or of v_2 in H). If $x, y \in \{v_1, v_2\}$, say $x = v_1$ and $y = v_2$, then there are two neighbors w_1 and z_1 of v_1, and two neighbors w_2 and z_2 of v_2 where $\{w_1, z_1\} \cap \{w_2, z_2\} = \varnothing$. Since $G - v$ is 2-connected, there are two internally disjoint z_1z_2-paths in $G - v$ and there are two internally disjoint w_1w_2-paths in $G - v$. If one of the z_1z_2-paths is disjoint from one of the w_1w_2-paths, then there are two disjoint internally disjoint v_1v_2-paths (giving, along with v_1ev_2, a total of three such paths).

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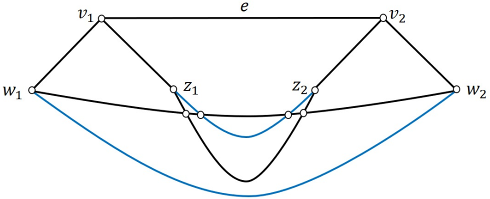

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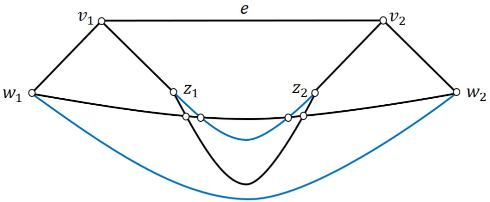

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Proof (continued). So we only need to consider the case where both internally disjoint z_1z_2-paths intersect both internally disjoint w_1w_2-paths. In Exercise 9.4.B it is to be shown that there are disjoint paths $P_{z_1w_2}$ (joining z_1 and w_2) and $P_{w_1z_2}$ (joining w_1 and z_2). Since $x = v_1$ is adjacent to w_1 and z_1, and $y = v_2$ is adjacent to w_2 and z_2 then there are two internally disjoint paths joining v_1 and v_2 through these points. So there are three internally disjoint v_1v_2-paths (including v_1ev_2). Therefore, H is 3-connected, as claimed. □