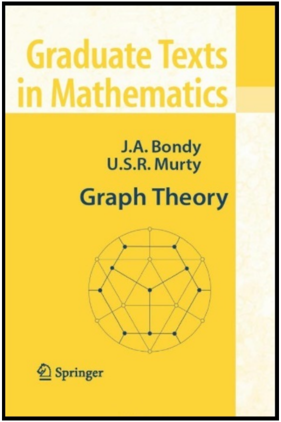
Graph Theory

Chapter 9. Connectivity

9.7. Chordal Graphs—Proofs of Theorems



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Theorem 9.19. Let G be a connected chordal graph which is not complete, and let S be a minimal vertex cut of G. Then S is a clique of G.

Proof. Let S be a minimal vertex cut of G. ASSUME S contains two nonadjacent vertices of G, say x and y. Let G_1 and G_2 be two components of $G - S$. Because S is a minimal cut, then in G both x and y are joined to vertices of both G_1 and G_2; if not, then either x or y could be dropped from S and we would have a smallest vertex cut of G, contradicting the minimality of S.

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Theorem 9.20

Theorem 9.20. Let G be a connected chordal graph, and let V_1 be a maximal clique of G. Then the maximal cliques of G can be arranged in a sequence $(V_1, v_2, \ldots, V_k)$ such that $V_j \cap \left(\cup_{i=1}^{j-1} V_j\right)$ is a clique of G for $2 \leq j \leq k$.

Proof. We give an inductive proof on k the number of maximal cliques of G. For the base case with $k = 1$, G is a complete graph and we take $V_1 = G$ and the claim holds trivially. For the inductive hypothesis, suppose the result holds for all graphs G with $k = \ell$ or less maximal cliques. Let G have $\ell + 1$ maximal cliques (where $\ell + 1 \geq 2$). Then G is not a complete graph and there are two nonadjacent vertices of G, which means that there are vertex cuts of G (notice the definition of a vertex cut requires the existence of two nonadjacent vertices). Let S be a minimal vertex cut of G. By Theorem 9.19, S is a clique of G.

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Theorem 9.20 (continued 1)

Proof (continued). Recall from Section 9.4. Three-Connected Graphs that an S-component of G is a subgraph H of G induced by the set of vertices $S \cup X$ where X is the vertex set of a component of $G - S$. Let H_i, for $1 \leq i \leq p$ be the S-components of G (so $p \geq 2$) and let Y_i be a maximal clique of H_i containing S for $1 \leq i \leq p$. By Exercise 9.7.1, the maximal cliques $H_1, H_2, \ldots, H_p$ are also maximal cliques are also maximal cliques of G and every maximal clique of G is a maximal clique of some H_i. So considering maximal cliques of G is equivalent to considering maximal degrees of the H_i's. Since V_1 is a maximal clique, we can suppose without loss of generality (or by reindexing) that V_1 is a maximal clique of H_1. The number of of maximal cliques of H_1 is at most ℓ (since the maximum cliques of the H_i's are maximum cliques of G and there are at least two H_i's), so by the induction hypothesis the maximal cliques of H_1 can be arranged in a sequence starting with V_1 and having the stated property in terms of intersections, unions, and cliques.

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Theorem 9.20. Let G be a connected chordal graph, and let V_1 be a maximal clique of G. Then the maximal cliques of G can be arranged in a sequence $(V_1, v_2, \ldots, V_k)$ such that $V_j \cap \left(\cup_{i=1}^{j-1} V_j \right)$ is a clique of G for $2 \leq j \leq k$.

Proof (continued). Likewise, for $2 \leq i \leq p$ the maximal cliques of H_i can be arranged in a suitable sequence starting with Y_i. Notice that when we union together all the maximal cliques of H_i we get H_i itself. Since the H_i's are S-components, then the intersection of H_i and H_j is S (for $i \neq j$), which is a clique. So if we concatenate all the sequences for $H_1, H_2, \ldots, H_p$ then we get a sequence of maximal cliques of G satisfying the stated property. So the result holds for $k = \ell + 1$ and, by mathematical induction, it holds for all graphs. □

Theorem 9.21

Theorem 9.21. Every chordal graph which is not complete has two nonadjacent simplicial vertices.

Proof. By Theorem 9.20, chordal graph G has a simplicial decomposition, say $(V_1, V_2, \ldots, V_k)$ where $k \geq 2$ since G is not complete. Let $x \in V_k \setminus \left(\cup_{i=1}^{k-1} V_i\right)$. In Exercise 7.9.A it is to be shown that x is a simplicial vertex. Notice that by Theorem 9.20, V_1 can be *any* maximal clique of G, so that a simplicial decomposition may start with any maximal clique (though we are not then free to choose the remaining maximum cliques in the sequence in any order we wish; we are only free to choose the first maximum clique in the sequence).

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Theorem 9.23

Theorem 9.23. A graph is chordal if and only if it is the intersection graph of a family of subtrees of a tree.

Proof. Let G be a chordal graph. By Theorem 9.20, G has a simplicial decomposition $(V_1, V_2, \ldots, V_k)$. We give an inductive proof on k, the number of maximal cliques of G. We'll show that G is the intersection graph of a family of subtrees $\mathcal{T} = \{T_v \mid v \in V\}$ of a tree T with vertex set $x_1, x_2, \ldots, x_k\}$ such that $x_i \in T_v$ for all $v \in V_i$. The converse that if G is the intersection graph of a family of subtrees of a tree, then G is chordal, is left as Exercise 9.7.4.

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Theorem 9.23. A graph is chordal if and only if it is the intersection graph of a family of subtrees of a tree.

Proof (continued). For the base case $k = 1$, G is a complete graph. Let T be a tree with vertex set $\{x_1\}$ and for each $v \in V(G)$ set $T_v = T$. Then for any $u, v \in V(G)$ we have $T_v \cap T_u = \{x_1\} \neq \varnothing$ so that the intersection graph of $(T, \mathcal{T})$ is a complete graph on $v(G)$ vertices, as claimed. For the induction hypothesis, suppose that every graph with $k = \ell$ maximal cliques is the intersection graph of a family of subtrees of a tree on ℓ vertices.

Let G be a chordal graph with $k = \ell + 1$ maximal degrees (where $k = \ell + 1 \geq 2$). Then by Theorem 9.20 G has a simplicial decomposition $(V_1, V_2, \ldots, V_{\ell+1})$. Consider the simplicial decomposition $(V_1, V_2, \ldots, V_\ell)$ and let graph $G' = (V', E')$ be the graph for which this is the simplicial decomposition (the vertex set V' is simply $\cup_{i=1}^{\ell} V_i$ and the edge set E' is the union of ℓ edge sets of complete graphs with vertex sets $(V_1, V_2, \ldots, V_\ell)$.

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Proof (continued). By the induction hypothesis, G' is the intersection graph of a family of subtrees $\mathcal{T} = \{T_v \mid v \in V'\}$ of a tree T' with vertex set $\{x_1, x_2, \ldots, x_{\ell+1}\}$. (We need to construct tree T and family of subtrees $\mathcal{T}$ for graph G.) Let V_j be a maximal clique of G' such that $V_j \cap V_{\ell+1} \neq \varnothing$ (such V_j exists since $\ell + 1 \geq 2$). We next form tree T from tree T' by adding a new vertex of T, $X_{\ell+1}$, adjacent to vertex x_j of T'. For $v \in V_j$ form tree T_v by adding $x_{\ell+1}$ to tree T'_v be joining it to x_j. For $v \in V' \setminus V_j$, set $T_v = T'_v$. Finally, for $v \in V_{\ell+1} \setminus V'$ set $T_v = \{x_{\ell+1}\}$, and define $\mathcal{T} = \{T_v \mid v \in V\}$. In Exercise 9.7.B, is is to be shown that the intersection graph of $(T, \mathcal{T})$ does, in fact, have the same adjacency structure as graph G. □