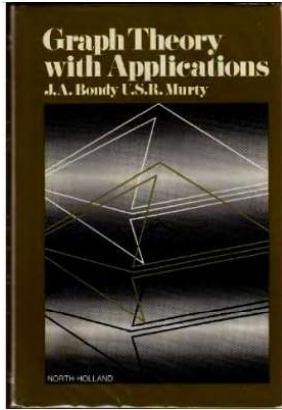


Graph Theory

Chapter 12. The Cycle Space and Bond Space

12.1. Circulations and Potential Differences—Proofs of Theorems



Theorem 12.1

Theorem 12.1. Let \mathbf{M} be the incidence matrix of a digraph D . Then \mathcal{B} is the row space of \mathbf{M} and \mathcal{C} is its orthogonal complement.

Proof. Let p be a real-valued function on the vertex set V of D and let $g = \delta p$ be the associated potential difference. Since for arc a (a link) with tail x and head y we have $\delta p(a) = p(x) - p(y)$, then for all $a \in A$ we have

$$g(a) = \delta p(a) = p(x) - p(y) = (0 + 0 + \dots + 0 + p(x) + 0 + \dots + 0) + (0 - 0 - \dots - 0 - p(y) - 0 - \dots - 0) = \sum_{v \in V} p(v) m_v(a), \quad (*)$$

because $m_v(a) = 1$ when v is the tail of a , $m_v(a) = -1$ when v is the head of a , and $m_v(a) = 0$ for all other v . Since row v of \mathbf{M} has entries $m_v(a)$ where a ranges over the arcs of D , then potential difference g is a linear combination of the rows of \mathbf{M} (as a varies over all m arcs, $g(a)$ is an m -tuple) where the v th row has coefficient $p(v)$.

Theorem 12.1 (continued 1)

Proof (continued). Conversely, any linear combination of the rows of \mathbf{M} , $\sum_{v \in V} b_v m_v(a)$, is a potential difference $g = \delta p$, as in (*), where $p(v) = b_v$ (the coefficients). This is because $m_x(a) = 1$ when x is the tail of a , $m_y(a) = -1$ when y is the head of a , and $m_v(a) = 0$ for all other v , so that $g(a) = b_x m_x(a) + b_y m_y(a) = p(x)(1) + p(y)(-1) = p(x) - p(y)$. Therefore the row space of incidence matrix \mathbf{M} is the bond space \mathcal{B} .

Next, let f be a function on A . Then f is a circulation in D if (by definition of "circulation") $f^-(v) = f^+(v)$ for all $v \in V$. For any $a \in A$ we have $m_x(a) = 1$ when x is the tail of a , $m_y(a) = -1$ when y is the head of a , and $m_v(a) = 0$ for all other v , so

$$m_v(a)f(a) = \begin{cases} 0 & \text{if } v \notin \{x, y\} \\ m_x(a)f(a) = f(a) & \text{if } v = x \\ m_y(a)f(a) = -f(a) & \text{if } v = y \end{cases}$$

Theorem 12.1 (continued 2)

Proof (continued). Recall that $f^+(v) = \sum_{a \in A, a \text{ is an arc from } v} f(a)$ and

$$f^-(v) = \sum_{a \in A, a \text{ is an arc to } v} f(a).$$

Summing $m_v(a)f(a)$ over all $a \in A$ (so that all arcs with v as the tail contribute to $f^+(v)$ and all arcs with v as the head contribute to $f^-(v)$), we have

$$\sum_{a \in A} m_v(a)f(a) = f^+(v) - f^-(v) = 0 \text{ for all } v \in V.$$

As a ranges over all arcs, $m_v(a)$ is the v th row of \mathbf{M} and $f(a)$ produces the vector $[f(a_1), f(a_2), \dots, f(a_m)]$ (say). So $\sum_{a \in A} m_v(a)f(a)$ is the inner product (or dot product) of the v th row vector with the vector $[f(a_1), f(a_2), \dots, f(a_m)]$. Since this inner product is 0 and $v \in V$ is arbitrary, then each element of the row space of \mathbf{M} (namely, the vectors in \mathcal{B} by the first part of this theorem) is perpendicular to each circulation (treated as a vector; that is, treated as an element of \mathcal{C}).

Theorem 12.1 (continued 3)

Theorem 12.1. Let \mathbf{M} be the incidence matrix of a digraph D . Then \mathcal{B} is the row space of \mathbf{M} and \mathcal{C} is its orthogonal complement.

Proof (continued). Conversely, any function f on A yields a vector $[f(a_1), f(a_2), \dots, f(a_m)]$ which satisfies $\sum_{a \in A} m_v(a)f(a) = 0$ for all $v \in V$ then, since $\sum_{a \in A} m_v(a)f(a) = f^+(v) - f^-(v)$ we must have $f^+(v) = f^-(v)$ for all $v \in V$. That is, f is a circulation. So the perp space of \mathcal{B} , \mathcal{B}^\perp , is \mathcal{C} , as claimed. \square

Note. The row vectors of \mathcal{B} and the vectors based on circulations all are vectors in \mathbb{R}^m where $m = |A|$. So that \mathcal{B} and \mathcal{C} are orthogonal complements in \mathbb{R}^m .

Lemma 12.2.1

Lemma 12.2.1. If f is a nonzero circulation (that is, f is not identically zero), then $\|f\|$ contains a cycle.

Proof. By the conservation condition of a circulation, $f^-(v) = f^+(v)$, and the fact that f is nonzero, we see that the support of f , $\|f\|$, cannot contain a vertex of degree one. So all vertices of $\|f\|$ must be of degree at least two. Then by Theorem 2.1 (of Bondy and Murty's graduate text), the underlying undirected graph of $\|f\|$ contains a cycle, as claimed. \square

Lemma 12.2.2

Lemma 12.2.2. If g is a nonzero potential difference (that is, g is not identically zero), then $\|g\|$ contains a bond.

Proof. Let $g = \delta p$ be a nonzero potential difference in digraph D . Since g is not identically zero, then $\|g\|$ contains some arc. Choose a vertex $u \in V$ which is incident with an arc of $\|g\|$ and set $U = \{v \in V \mid p(v) = p(u)\}$. Notice that $\bar{U} = V \setminus U$ consists of vertices $v \in V$ such that $p(v) \neq p(u)$, so all arcs in the induced digraph $[U, \bar{U}]$ have nonzero potential difference and hence are in $\|g\|$. That is, $\|g\| \supseteq [U, \bar{U}]$. Now u is chosen such that it is incident to an arc of $\|g\|$, so $[U, \bar{U}]$ is nonempty. Therefore $[U, \bar{U}]$ is an edge cut of the (underlying) graph of D . Since a bond is a minimal edge cut, then $[U, \bar{U}]$ contains a bond and because, $\|g\| \supseteq [U, \bar{U}]$, $\|g\|$ contains a bond, as claimed. \square

Theorem 12.2

Theorem 12.2. Let \mathbf{B} and \mathbf{C} be basis matrices of \mathcal{B} and \mathcal{C} , respectively. Then for any $S \subseteq A$:

- (i) the columns of $\mathbf{B}|_S$ are linearly independent if and only if S is acyclic, and
- (ii) the columns of $\mathbf{C}|_S$ are linearly independent if and only if S contains no bond.

Proof. Denote the column of \mathbf{B} corresponding to arc a by $\mathbf{B}(a)$. The columns of $\mathbf{B}|_S$ are linearly dependent if and only if there exists a function f on A such that $f(a) \neq 0$ for some $a \in S$, $f(a) = 0$ for all $a \notin S$, and $\sum_{a \in A} f(a)\mathbf{B}(a) = \mathbf{0}$ (that is, the coefficients in a linear combination of the columns of $\mathbf{B}|_S$ are not all 0, yet the linear combination is $\mathbf{0}$). Therefore the columns of $\mathbf{B}|_S$ are linearly dependent if and only if there exists a nonzero circulation f (that is, f is not identically zero) such that $\|f\| \subseteq S$.

Theorem 12.2 (continued 1)

Proof (continued). If there is such a circulation f , then by Lemma 12.2.1 $\|f\|$, and therefore S , contains a cycle. That is, if $\mathbf{B}|_S$ is linearly dependent then S contains a cycle. In other words (considering the contrapositive), if S is acyclic then $\mathbf{B}|_S$ is linearly independent, as claimed.

On the other hand, suppose S contains a cycle C (that is, suppose the arcs of S induce a sub-digraph of D whose underlying graph contains a cycle); i.e., suppose S is not acyclic. Consider the circulation on D of Note 12.1.B, f_C . Then $f_C(a) = \pm 1$ for each $a \in C$ (hence f_C is not identically zero) so that the support is $\|f_C\| = C \subseteq S$. As observed above, this implies that the columns of $\mathbf{B}|_S$ are linearly dependent. In other words (considering the contrapositive), if $\mathbf{B}|_S$ is linearly independent then S is acyclic, as claimed. We have shown that the columns of $\mathbf{B}|_S$ are linearly independent if and only if S is acyclic, establishing (i).

Corollary 12.2

Corollary 12.2. Let D be a digraph. The dimensions of the bond space \mathcal{B} and the cycle space \mathcal{C} are given by $\dim(\mathcal{B}) = \nu - \omega$ and $\dim(\mathcal{C}) = \varepsilon - \nu + \omega$, where ν is the number of vertices of D , ε is the number of arcs of D , and ω is the number of connected components of D .

Proof. Consider the basis matrix \mathbf{B} of \mathcal{B} . By Theorem 12.2, $\text{rank}(\mathbf{B}) = \max\{|S| \mid S \subset A, S \text{ is acyclic}\}$. A maximal acyclic subgraph of a connected graph is a spanning tree, and a maximal acyclic subgraph of a graph is a spanning forest (consisting of a spanning tree of each connected component). By Exercise 2.2.4 of Bondy and Murty's *Graph Theory with Applications* or by Exercise 4.1.1 of Bondy and Murty's graduate level *Graph Theory*, a maximal forest has $\nu - \omega$ arcs. So $\dim(\mathcal{B}) = \text{rank}(\mathbf{B})$, as claimed.

Theorem 12.2 (continued 2)

Theorem 12.2. Let \mathbf{B} and \mathbf{C} be basis matrices of \mathcal{B} and \mathcal{C} , respectively. Then for any $S \subseteq A$:

- (i) the columns of $\mathbf{B}|_S$ are linearly independent if and only if S is acyclic, and
- (ii) the columns of $\mathbf{C}|_S$ are linearly independent if and only if S contains no bond.

Proof (continued). The proof of (ii) is similar to the proof of (i), except that we use Lemma 12.2.2 in place of Lemma 12.2.1 and we use potential difference g_B of Note 12.1.D in place of the circulation f_C of Note 12.1.B. □

Corollary 12.2 (continued)

Corollary 12.2. Let D be a digraph. The dimensions of the bond space \mathcal{B} and the cycle space \mathcal{C} are given by $\dim(\mathcal{B}) = \nu - \omega$ and $\dim(\mathcal{C}) = \varepsilon - \nu + \omega$, where ν is the number of vertices of D , ε is the number of arcs of D , and ω is the number of connected components of D .

Proof (continued). By Theorem 12.1, the bond space \mathcal{B} and the cycle space \mathcal{C} of digraph D are orthogonal complements of each other in the space \mathbb{R}^ε (see the note at the end of the proof of Theorem 12.1). So $\dim(\mathcal{C}) = \varepsilon - \dim(\mathcal{B}) = \varepsilon - \nu + \omega$, as claimed. □