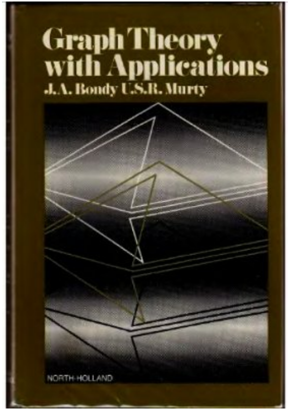
# Graph Theory

**Chapter 12. The Cycle Space and Bond Space**

12.1. Circulations and Potential Differences—Proofs of Theorems



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**Proof.** Let $p$ be a real-valued function on the vertex set $V$ of $D$ and let $g = \delta p$ be the associated potential difference. Since for arc $a$ (a link) with tail $x$ and head $y$ we have $\delta p(a) = p(x) - p(y)$, then for all $a \in A$ we have

$$g(a) = \delta p(a) = p(x) - p(y) = (0 + 0 + \cdots + 0 + p(x) + 0 + \cdots + 0)$$

$$+(0 - 0 - \cdots - 0 - p(y) - 0 - \cdots - 0) = \sum_{v \in V} p(v) m_v(a), \qquad (*)$$

because $m_v(a) = 1$ when $v$ is the tail of $a$, $m_v(a) = -1$ when $v$ is the head of $a$, and $m_v(a) = 0$ for all other $v$.

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**Proof (continued).** Conversely, any linear combination of the rows of $\mathbf{M}$, $\sum_{v \in V} b_v m_v(a)$, is a potential difference $g = \delta p$, as in $(*)$, where $p(v) = b_v$ (the coefficients). This is because $m_x(a) = 1$ when $x$ is the tail of $a$, $m_y(a) = -1$ when $y$ is the head of $a$, and $m_v(a) = 0$ for all other $v$, so that $g(a) = b_x m_x(a) + b_y m_y(a) = p(x)(1) + p(y)(-1) = p(x) - p(y)$. Therefore the row space of incidence matrix $\mathbf{M}$ is the bond space $\mathcal{B}$.

Next, let $f$ be a function on $A$. Then $f$ is a circulation in $D$ if (by definition of "circulation") $f^-(v) = f^+(v)$ for all $v \in V$. For any $a \in A$ we have $m_x(a) = 1$ when $x$ is the tail of $a$, $m_y(a) = -1$ when $y$ is the head of $a$, and $m_v(a) = 0$ for all other $v$, so

$$m_v(a)f(a) = \begin{cases} 0 & \text{if } v \notin \{x, y\} \\ m_x(a)f(a) = f(a) & \text{if } v = x \\ m_y(a)f(a) = -f(a) & \text{if } v = y \end{cases}$$

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# Theorem 12.1 (continued 2)

**Proof (continued).** Recall that $f^+(v) = \sum_{a \in A,\ a \text{ is an arc from } v} f(a)$ and $f^-(v) = \sum_{a \in A,\ a \text{ is an arc to } v} f(a)$. Summing $m_v(a)f(a)$ over all $a \in A$ (so that all arcs with $v$ as the tail contribute to $f^+(v)$ and all arcs with $v$ as the head contribute to $f^-(v)$), we have

$\sum_{a \in A} m_v(a)f(a) = f^+(v) - f^-(v) = 0$ for all $v \in V$. As $a$ ranges over all arcs, $m_v(a)$ is the $v$th row of $\mathbf{M}$ and $f(a)$ produces the vector $[f(a_1), f(a_2), \ldots, f(a_m)]$ (say). So $\sum_{a \in A} m_v(a)f(a)$ is the inner product (or dot product) of the $v$th row vector with the vector $[f(a_1), f(a_2), \ldots, f(a_m)]$. Since this inner product is 0 and $v \in V$ is arbitrary, then each element of the row space of $\mathbf{M}$ (namely, the vectors in $\mathcal{B}$ by the first part of this theorem) is perpendicular to each circulation (treated as a vector; that is, treated as an element of $\mathcal{C}$.

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**Theorem 12.1.** Let $\mathbf{M}$ be the incidence matrix of a digraph $D$. Then $\mathcal{B}$ is the row space of $\mathbf{M}$ and $\mathcal{C}$ is its orthogonal complement.

**Proof (continued).** Conversely, any function $f$ on $A$ yields a vector $[f(a_1), f(a_2), \ldots, f(a_m)]$ which satisfies $\displaystyle\sum_{a \in A} m_v(a) f(a) = 0$ for all $v \in V$ then, since $\sum_{a \in A} m_v(a) f(a) = f^+(v) - f^-(v)$ we must have $f^+(v) = f^-(v)$ for all $v \in V$. That is, $f$ is a circulation. So the perp space of $\mathcal{B}$, $\mathcal{B}^\perp$, is $\mathcal{C}$, as claimed. □

**Note.** The row vectors of $\mathcal{B}$ and the vectors based on circulations all are vectors in $\mathbb{R}^m$ where $m = |A|$. So that $\mathcal{B}$ and $\mathcal{C}$ are orthogonal complements in $\mathbb{R}^m$.

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# Lemma 12.2.1

**Lemma 12.2.1.** If $f$ is a nonzero circulation (that is, $f$ is not identically zero), then $\|f\|$ contains a cycle.

**Proof.** By the conservation condition of a circulation, $f^{-}(v) = f^{+}(v)$, and the fact that $f$ is nonzero, we see that the support of $f$, $\|f\|$, cannot contain a vertex of degree one. So all vertices of $\|f\|$ must be of degree at least two. Then by Theorem 2.1 (of Bondy and Murty's graduate text), the underlying undirected graph of $\|f\|$ contains a cycle, as claimed. □

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# Lemma 12.2.2

**Lemma 12.2.2.** If $g$ is a nonzero potential difference (that is, $g$ is not identically zero), then $\|g\|$ contains a bond.

**Proof.** Let $g = \delta p$ be a nonzero potential difference in digraph $D$. Since $g$ is not identically zero, then $\|g\|$ contains some arc. Choose a vertex $u \in V$ which is incident with an arc of $\|g\|$ and set $U = \{v \in V \mid p(v) = p(u)\}$. Notice that $\overline{U} = V \setminus U$ consists of vertices $v \in V$ such that $p(v) \neq p(u)$, so all arcs in the induced digraph $[U, \overline{U}]$ have nonzero potential difference and hence at in $\|g\|$. That is, $\|g\| \supseteq [U, \overline{U}]$.

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# Theorem 12.2

**Theorem 12.2.** Let $\mathbf{B}$ and $\mathbf{C}$ be basis matrices of $\mathcal{B}$ and $\mathcal{C}$, respectively. Then for any $S \subseteq A$:

(i) the columns of $\mathbf{B}|S$ are linearly independent if and only if $S$ is acyclic, and

(ii) the columns of $\mathbf{C}|S$ are linearly independent if and only if $S$ contains no bond.

**Proof.** Denote the column of $\mathbf{B}$ corresponding to arc $a$ by $\mathbf{B}(a)$. The columns of $\mathbf{B}|S$ are linearly dependent if and only if there exists a function $f$ on $A$ such that $f(a) \neq 0$ for some $a \in S$, $f(a) = 0$ for all $a \notin S$, and $\sum_{a \in A} f(a)\mathbf{B}(a) = \mathbf{0}$ (that is, the coefficients in a linear combination of the columns of $\mathbf{B}|S$ are not all 0, yet the linear combination is $\mathbf{0}$).

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# Theorem 12.2 (continued 1)

**Proof (continued).** If there is such a circulation $f$, then by Lemma 12.2.1 $\|f\|$, and therefore $S$, contains a cycle. That is, if $\mathbf{B}|S$ is linearly dependent then $S$ contains a cycle. In other words (considering the contrapositive), if $S$ is acyclic then $\mathbf{B}|S$ is linearly independent, as claimed.

On the other hand, suppose $S$ contains a cycle $C$ (that is, suppose the arcs of $S$ induce a sub-digraph of $D$ whose underlying graph contains a cycle); i.e., suppose $S$ is not acyclic. Consider the circulation on $D$ of Note 12.1.B, $f_C$. Then $f_C(a) = \pm 1$ for each $a \in C$ (hence $f_C$ is not identically zero) so that the support is $\|f_C\| = C \subseteq S$. As observed above, this implies that the columns of $\mathbf{B}|S$ are linearly dependent. In other words (considering the contrapositive), if $\mathbf{B}|S$ is linearly independent then $S$ is acyclic, as claimed. We have shown that the columns of $\mathbf{B}|S$ are linearly independent if and only if $S$ is acyclic, establishing (i).

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# Theorem 12.2 (continued 2)

**Theorem 12.2.** Let **B** and **C** be basis matrices of $\mathcal{B}$ and $\mathcal{C}$, respectively. Then for any $S \subseteq A$:

(i) the columns of $\mathbf{B}|S$ are linearly independent if and only if $S$ is acyclic, and

(ii) the columns of $\mathbf{C}|S$ are linearly independent if and only if $S$ contains no bond.

**Proof (continued).** The proof of (ii) is similar to the proof of (i), except that we use Lemma 12.2.2 in place of Lemma 12.2.1 and we use potential difference $g_B$ of Note 12.1.D in place of the circulation $f_C$ of Note 12.1.B. □

# Corollary 12.2

**Corollary 12.2.** Let $D$ be a digraph. The dimensions of the bond space $\mathcal{B}$ and the cycle space $\mathcal{C}$ are given by $\dim(\mathcal{B}) = \nu - \omega$ and $\dim(\mathcal{C}) = \varepsilon - \nu + \omega$, where $\nu$ is the number of vertices of $D$, $\varepsilon$ is the number of arcs of $D$, and $\omega$ is the number of connected components of $D$.

**Proof.** Consider the basis matrix $\mathbf{B}$ of $\mathcal{B}$. By Theorem 12.2, $\text{rank}(\mathbf{B}) = \max\{|S| \mid S \subset A, S \text{ is acyclic}\}$. A maximal acyclic subgraph of a connected graph is a spanning tree, and a maximal acyclic subgraph of a graph is a spanning forest (consisting of a spanning tree of each connected component). By Exercise 2.2.4 of Bondy and Murty's *Graph Theory with Applications* or by Exercise 4.1.1 of Bondy and Murty's graduate level *Graph Theory*, a maximal forest has $\nu - \omega$ arcs. So $\dim(\mathcal{B}) = \text{rank}(\mathbf{B})$, as claimed.

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**Proof.** Consider the basis matrix $\mathbf{B}$ of $\mathcal{B}$. By Theorem 12.2, $\text{rank}(\mathbf{B}) = \max\{|S| \mid S \subset A, S \text{ is acyclic}\}$. A maximal acyclic subgraph of a connected graph is a spanning tree, and a maximal acyclic subgraph of a graph is a spanning forest (consisting of a spanning tree of each connected component). By Exercise 2.2.4 of Bondy and Murty's *Graph Theory with Applications* or by Exercise 4.1.1 of Bondy and Murty's graduate level *Graph Theory*, a maximal forest has $\nu - \omega$ arcs. So $\dim(\mathcal{B}) = \text{rank}(\mathbf{B})$, as claimed.

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**Proof (continued).** By Theorem 12.1, the bond space $\mathcal{B}$ and the cycle space $\mathcal{C}$ of digraph $D$ are orthogonal complements of each other in the space $\mathbb{R}^\varepsilon$ (see the note at the end of the proof of Theorem 12.1). So $\dim(\mathcal{C}) = \varepsilon - \dim(\mathcal{B}) = \varepsilon - \nu + \omega$, as claimed. □