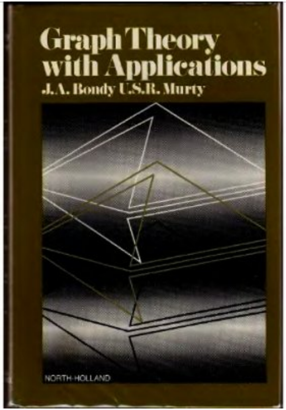
Graph Theory

Chapter 12. The Cycle Space and Bond Space

12.2. The Number of Spanning Trees—Proofs of Theorems



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Theorem 12.3

Theorem 12.3. Let G be a graph and T a spanning tree of G. Let D be any orientation of G and let $\mathbf{B}$ be the basis matrix of the bond space $\mathcal{B}$ corresponding to T. Then $\mathbf{B}$ is a unimodular matrix.

Proof. Let $\mathbf{P}$ be a submatrix of $\mathbf{B}$ is size $(v-1)\times(v-1)$. Suppose that $\mathbf{P} = \mathbf{B}|T_1$. We may assume that (the arcs of) T_1 is (form) a spanning tree of D, since if it has $v-1$ arcs and is not a spanning tree then if contains a cycle and then by Theorem 12.2(i) the columns of $\mathbf{P} = \mathbf{B}|T_1$ are linearly dependent and $\det(\mathbf{P}) = 0$. Let $\mathbf{B}_1$ denote the basis matrix of the bond space $\mathcal{B}$ corresponding to tree T_1. Then by Exercise 12.1.2(b) of Bondy and Murty's *Graph Theory with Applications*, $(\mathbf{B}|T_1)\mathbf{B}_1 = \mathbf{B}$. Restricting both sides of this equation to T (that is, to the columns corresponding to the arcs in tree T), we obtain

$$((\mathbf{B}|T_1)\mathbf{B_1})|T) = (\mathbf{B}|T_1)(\mathbf{B}_1|T) = \mathbf{B}|T.$$

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Proof (continued). ...

$$((\mathbf{B}|T_1)\mathbf{B_1})|T) = (\mathbf{B}|T_1)(\mathbf{B}_1|T) = \mathbf{B}|T.$$

Now $\mathbf{B}|T$ is an identity matrix by Note 12.1.F, so $\det(\mathbf{B}|T) = 1$. Since the determinant of the product of two square matrices is the product of the determinants (see my online Linear Algebra [MATH 2010] notes on Section 4.2. The Determinant of a Square Matrix; notice Theorem 4.4, "The Multiplicative Property") then we have $\det(\mathbf{B}|T_1)\det(\mathbf{B}_1|T) = 1$. Both matrices $\mathbf{B}|T_1$ and $\mathbf{B}_1|T$ have only integer entries, so the determinants are themselves integers. This implies $\det(\mathbf{B}|T_1) = \pm 1$, as claimed. □

Theorem 12.4

Theorem 12.4. Let G be a graph and T a spanning tree of G. Let D be any orientation of G and let $\mathbf{B}$ be the basis matrix of the bond space $\mathcal{B}$ corresponding to T. The number of spanning trees of G is $\tau(G) = \det(\mathbf{B}\mathbf{B}')$, where $\mathbf{B}'$ is the transpose of $\mathbf{B}$.

Proof. By Hadley's Theorem and Equation (12.7) of Note 12.2.B, we have

$$\det(\mathbf{B}\mathbf{B}') = \sum_{S \subseteq A,\, |S| = \nu - 1} (\det(\mathbf{B}|S))^2.$$

By Theorem 12.2(i), the columns of $\mathbf{B}|S$ are linearly independent if and only if S is acyclic (that is, if and only if the arcs of S determine an orientation of a spanning tree of G).

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Theorem 12.4 (continued)

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Proof (continued). The columns of a square matrix are linearly independent if and only of the determinant of the matrix is nonzero (see my online notes for Linear Algebra [MATH 2010] on Section 1.5. Inverses of Matrices, and Linear Systems, Theorem 1.12 "Conditions for A^{-1} to Exist" and Section 4.2. The Determinant of a Square Matrix, Theorem 4.3, "Determinant Criterion for Invertibility") so the number of nonzero terms in the sum is equal to the number of spanning trees, $\tau(G)$. By Theorem 12.3, $\mathbf{B}$ is unimodular, so each nonzero term in the sum is 1. Therefore, the number of spanning trees in G is $\tau(G) = \det(\mathbf{BB}')$, as claimed. □

Corollary 12.4

Corollary 12.4. Let G be a graph and T a spanning tree of G. Let D be any orientation of G, let $\mathbf{B}$ be the basis matrix of the bond space $\mathcal{B}$ corresponding to T, and let $\mathbf{C}$ be the basis matrix of the cycle space $\mathcal{C}$ corresponding to T. The number of spanning trees of G is $\tau(G) = \pm\det\begin{bmatrix} \mathbf{B} \\ \mathbf{C} \end{bmatrix}$.

Proof. By Theorem 12.4 and Note 12.2.C we have $\tau(G) = \det(\mathbf{BB}')\det(\mathbf{CC}')$. By Theorem 3.1.G of Note 12.2.C we now have

$$(\tau(G))^2 = \det(\mathbf{BB}')\det(\mathbf{CC}') = \det\begin{bmatrix} \mathbf{BB}' & \mathbf{0} \\ \mathbf{0} & \mathbf{CC}' \end{bmatrix}.$$

Since the bond space $\mathcal{B}$ and the cycle space $\mathcal{C}$ are orthogonal complements, then $\mathbf{BC}' = \mathbf{CB}' = \mathbf{0}$ (for example, notice that the (i,j) entry of $\mathbf{BC}'$ is the dot product of the i row of $\mathbf{B}$ with the jth column of $\mathbf{C}'$).

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Proof. By Theorem 12.4 and Note 12.2.C we have
$\tau(G) = \det(\mathbf{B}\mathbf{B}')\det(\mathbf{C}\mathbf{C}')$. By Theorem 3.1.G of Note 12.2.C we now have

$$(\tau(G))^2 = \det(\mathbf{B}\mathbf{B}')\det(\mathbf{C}\mathbf{C}') = \det\begin{bmatrix} \mathbf{B}\mathbf{B}' & \mathbf{0} \\ \mathbf{0} & \mathbf{C}\mathbf{C}' \end{bmatrix}.$$

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Corollary 12.4 (continued)

Proof (continued). Thus by Theorem 3.2.2 of Note 12.2.C,

$$(\tau(G))^2 = \det\begin{bmatrix} \mathbf{BB}' & \mathbf{0} \\ \mathbf{0} & \mathbf{CC}' \end{bmatrix} = \begin{bmatrix} \mathbf{BB}' & \mathbf{BC}' \\ \mathbf{CB}' & \mathbf{CC}' \end{bmatrix} = \det\left(\begin{bmatrix} \mathbf{B} \\ \mathbf{C} \end{bmatrix}\begin{bmatrix} \mathbf{B}' & \mathbf{C}' \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} \mathbf{B} \\ \mathbf{C} \end{bmatrix}\det\begin{bmatrix} \mathbf{B}' & \mathbf{C}' \end{bmatrix} = \left(\det\begin{bmatrix} \mathbf{B} \\ \mathbf{C} \end{bmatrix}\right)^2,$$

since $\det(\mathbf{A}) = \det(\mathbf{A}')$ by Theorem 4.2.A, "Properties of the Determinant" in my online Linear Algebra (MATH 2010) notes on Section 4.2. The Determinant of a Square Matrix. Notice that $\mathbf{B}$ is $(\nu - 1) \times \varepsilon$ and $\mathbf{C}$ is $(\varepsilon - \nu + 1) \times \varepsilon$, so that $\begin{bmatrix} \mathbf{B} \\ \mathbf{C} \end{bmatrix}$ is $\varepsilon \times \varepsilon$ (square), as needed. Taking square roots gives the desired result. □