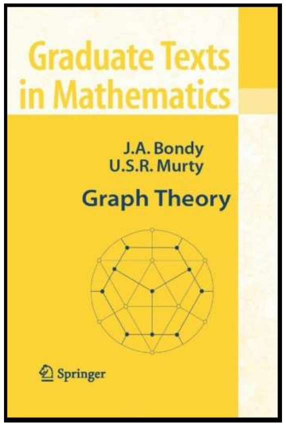
Graph Theory

Chapter 2. Subgraphs

Supplement. Graph Decompositions: Triple Systems—Proofs of Theorems



Lemma T.1

Lemma T.1. If $STS(n)$ exists then $n \equiv 1$ or $3 \pmod 6$.

Proof. The graph K_n has $\binom{n}{2} = \frac{n(n-1)}{2}$ total edges and a 3-cycle has 3 edges. Now the edge sets of the 3-cycles in a Steiner triple system partition the edge set of K_n, so it is necessary that 3 divides $\frac{n(n-1)}{2}$; that is, we must have $n(n-1) \equiv 0 \pmod 6$. So we must have $n \equiv 0$ or $1 \pmod 3$.

Next, the degree of each vertex in K_n is $n-1$ and the degree of each vertex of a 3-cycle is 2, so we must also have $n-1$ even, or n odd. Therefore, it is necessary that $n \equiv 1$ or $3 \pmod 6$. □

Lemma T.2

Lemma T.2. An (A,k)-system exists if and only if $k \equiv 0$ or $1 \pmod 4$.

Proof. If an (A,k)-system exists, then there is a partition of the set $\{1,2,\ldots,2k\}$ into distinct pairs (a_r,b_r) such that $b_r = a_r + r$ (or, equivalently, $b_r - a_r = r$) for $r = 1,2,\ldots,k$. So we must have $\sum_{r=1}^{k} b_r - \sum_{r=1}^{k} a_r = \sum_{r=1}^{k} r = \frac{k(k+1)}{2}$. Also, since the (a_r,b_r) partition $\{1,2,\ldots,2k\}$, then $\sum_{r=1}^{k} a_r + \sum_{r=1}^{k} b_r = \sum_{r=1}^{2k} r = \frac{2k(2k+1)}{2}$. Adding these equations, we must have $2\sum_{r=1}^{k} b_r = \frac{k(k+1)}{2} + \frac{2k(2k+1)}{2} = \frac{k((k+1)+2(2k+1))}{2} = \frac{k(5k+3)}{2}$, or $\sum_{r=1}^{k} b_r = \frac{k(5k+3)}{4}$, so that $k \equiv 0$ or $1 \pmod 4$ is necessary.

Lemma T.2 (continued 1)

Proof (continued). Now suppose $k \equiv 0 \pmod 4$, say $k = 4\ell$. If $k = 4$ and $\ell = 1$, consider the pairs $(1,2)$, $(5,7)$, $(3,6)$, and $(4,8)$. If $k \geq 8$ (and so $\ell \geq 2$) then consider the pairs

$$\begin{aligned} (4\ell+i-1, 8\ell-i+1) &\quad \text{for } i \in \{1,2,\ldots,2\ell\}, \\ (i, 4\ell-i-1) &\quad \text{for } i \in \{1,2,\ldots,\ell-1\}, \\ (\ell+i+1, 3\ell-i) &\quad \text{for } i \in \{1,2,\ldots,\ell-2\}, \\ (\ell,\ell+1), (2\ell, 4\ell-1), &\text{ and } (2\ell+1, 6\ell). \end{aligned}$$

We now check these pairs to insure that they are a (A,k)-system. In the following table we list the numbers covered by the pairs:

Pairs (a_r,b_r)	Range of a_r	Range of b_r
$(4\ell+i-1, 8\ell-i+1)$	$4\ell, 4\ell+1, \ldots, 6\ell-1$	$6\ell+1, 6\ell+2, \ldots, 8\ell$
$(i, 4\ell-i-1)$	$1, 2, \ldots, \ell-1$	$3\ell, 3\ell+1, \ldots, 4\ell-2$
$(\ell+i+1, 3\ell-i)$	$\ell+2, \ell+3, \ldots, 2\ell-1$	$2\ell+2, 2\ell+3, \ldots, 3\ell-1$
$(\ell, \ell+1)$	ℓ	$\ell+1$
$(2\ell, 4\ell-1)$	2ℓ	$4\ell-1$
$(2\ell+1, 6\ell)$	$2\ell+1$	6ℓ

Lemma T.2 (continued 2)

Proof (continued).

$$\begin{aligned} (4\ell+i-1, 8\ell-i+1) \quad & \text{for } i \in \{1,2,\ldots,2\ell\}, \\ (i, 4\ell-i-1) \quad & \text{for } i \in \{1,2,\ldots,\ell-1\}, \\ (\ell+i+1, 3\ell-i) \quad & \text{for } i \in \{1,2,\ldots,\ell-2\}, \\ (\ell,\ell+1), (2\ell, 4\ell-1), & \text{ and } (2\ell+1, 6\ell). \end{aligned}$$

We now check these pairs to insure that they are a (A, k)-system. In the following table we list the values of $b_r - a_r$:

Pairs (a_r, b_r)	$b_r - a_r$	Range of $b_r - a_r$
$(4\ell+i-1, 8\ell-i+1)$	$4\ell-2i+2$	$2, 4, \ldots, 4\ell$ even
$(i, 4\ell-i-1)$	$4\ell-2i-1$	$2\ell+1, 2\ell+3, \ldots, 4\ell-3$ odd
$(\ell+i+1, 3\ell-i)$	$2\ell-2i-1$	$3, 5, \ldots, 2\ell-3$
$(\ell, \ell+1)$	1	1
$(2\ell, 4\ell-1)$	$2\ell-1$	$2\ell-1$
$(2\ell+1, 6\ell)$	$4\ell-1$	$4\ell-1$

Lemma T.2 (continued 3)

Proof (continued). Now suppose $k \equiv 1 \pmod 4$, say $k = 4\ell+1$. If $k = 1$ and $\ell = 0$, consider the pair $(1, 2)$. If $k = 5$ and $\ell = 1$, consider the pairs $(2, 3)$, $(8, 10)$, $(4, 7)$, $(5, 9)$, and $(1, 6)$. If $k \geq 9$ (and so $\ell \geq 2$) then consider the pairs

$$\begin{aligned} (4\ell+i+1, 8\ell-i+3) \quad & \text{for } i \in \{1,2,\ldots,2\ell\}, \\ (i, 4\ell-i-1) \quad & \text{for } i \in \{1,2,\ldots,\ell\}, \\ (\ell+i+2, 3\ell-i+1) \quad & \text{for } i \in \{1,2,\ldots,\ell-2\}, \\ (\ell+1,\ell+2), (2\ell+1, 6\ell+2), & \text{ and } (2\ell+2, 4\ell+1). \end{aligned}$$

We now check these pairs to insure that they are a (A, k)-system. In the following table we list the numbers covered by the pairs:

Pairs (a_r, b_r)	Range of a_r	Range of b_r
$(4\ell+i+1, 8\ell-i+3)$	$4\ell+2, 4\ell+3, \ldots, 6\ell+1$	$6\ell+3, 6\ell+4, \ldots, 8\ell+2$
$(i, 4\ell-i+1)$	$1, 2, \ldots, \ell$	$3\ell+1, 3\ell+2, \ldots, 4\ell$
$(\ell+i+2, 3\ell-i+1)$	$\ell+3, \ell+4, \ldots, 2\ell$	$2\ell+3, 2\ell+4, \ldots, 3\ell$
$(\ell+1, \ell+2)$	$\ell+1$	$\ell+2$
$(2\ell+1, 6\ell+2)$	$2\ell+1$	$6\ell+2$
$(2\ell+2, 4\ell+1)$	$2\ell+2$	$4\ell+1$

Lemma T.2 (continued 4)

Proof (continued).

$$\begin{aligned} (4\ell+i+1, 8\ell-i+3) \quad & \text{for } i \in \{1,2,\ldots,2\ell\}, \\ (i, 4\ell-i+1) \quad & \text{for } i \in \{1,2,\ldots,\ell\}, \\ (\ell+i+2, 3\ell-i+1) \quad & \text{for } i \in \{1,2,\ldots,\ell-2\}, \\ (\ell+1,\ell+2), (2\ell+1, 6\ell+2), & \text{ and } (2\ell+2, 4\ell+1). \end{aligned}$$

We now check these pairs to insure that they are a (A, k)-system. In the following table we list the values of $b_r - a_r$:

Pairs (a_r, b_r)	$b_r - a_r$	Range of $b_r - a_r$
$(4\ell+i+1, 8\ell-i+3)$	$4\ell-2i+2$	$2, 4, \ldots, 4\ell$ even
$(i, 4\ell-i+1)$	$4\ell-2i+1$	$2\ell+1, 2\ell+3, \ldots, 4\ell-1$ odd
$(\ell+i+2, 3\ell-i+1)$	$2\ell-2i-1$	$3, 5, \ldots, 2\ell-3$
$(\ell+1, \ell+2)$	1	1
$(2\ell+1, 6\ell+2)$	$4\ell+1$	$4\ell+1$
$(2\ell+2, 4\ell+1)$	$2\ell-1$	$2\ell-1$

□

Lemma T.3

Lemma T.3. A (B, k)-system exists if and only if $k \equiv 2$ or $3 \pmod 4$.

Proof. If an (B, k)-system exists, then there is a partition of the set $\{1, 2, \ldots, 2k-1, 2k+1\}$ into distinct pairs (a_r, b_r) such that $b_r = a_r + r$ (or, equivalently, $b_r - a_r = r$) for $r = 1, 2, \ldots, k$. So we must have $\sum_{r=1}^k b_r - \sum_{r=1}^k a_r = \sum_{r=1}^k r = \frac{k(k+1)}{2}$. Also, since the (a_r, b_r) partition $\{1, 2, \ldots, 2k-1, 2k+1\}$, then $\sum_{r=1}^k a_r + \sum_{r=1}^k b_r = \sum_{r=1}^{2k-1} r + (2k+1) = \frac{(2k-1)(2k)}{2} + (2k+1) = (2k-1)k + (2k+1) = 2k^2 + k + 1$. Adding these equations, we must have $2\sum_{r=1}^k b_r = \frac{k(k+1)}{2} + (2k^2+k+1) = \frac{k(k+1) + 2(2k^2+k+1)}{2} = \frac{k^2+k+4k^2+2k+2}{2} = \frac{5k^2+3k+2}{2}$,

Lemma T.3 (continued 1)

Proof (continued). ...or $\sum_{r=1}^{k} b_r = \frac{5k^2+3k+2}{4}$. So we need $5k^2+3k+2 \equiv 0 \pmod 4$, and hence $k \equiv 2$ or $3 \pmod 4$ is necessary.

Now suppose $k \equiv 2 \pmod 4$, say $k = 4\ell+2$. If $k = 2$ and $\ell = 0$, consider the pairs $(1,2)$ and $(3,5)$. If $k \geq 6$ (and so $\ell \geq 1$) then consider the pairs

$$\begin{array}{rl} (i, 4\ell-i+2) & \text{for } i \in \{1,2,\ldots,2\ell\}, \\ (4\ell+i+3, 8\ell-i+4) & \text{for } i \in \{1,2,\ldots,\ell-1\}, \\ (5\ell+i+2, 7\ell-i+3) & \text{for } i \in \{1,2,\ldots,\ell-1\}, \end{array}$$
$$(2\ell+1, 6\ell+2), (4\ell+2, 6\ell+3), (4\ell+3, 8\ell+5) \text{ and } (7\ell+3, 7\ell+4).$$

Lemma T.3 (continued 2)

Proof (continued)....

$$\begin{array}{rl} (i, 4\ell-i+2) & \text{for } i \in \{1,2,\ldots,2\ell\}, \\ (4\ell+i+3, 8\ell-i+4) & \text{for } i \in \{1,2,\ldots,\ell-1\}, \\ (5\ell+i+2, 7\ell-i+3) & \text{for } i \in \{1,2,\ldots,\ell-1\}, \end{array}$$
$$(2\ell+1, 6\ell+2), (4\ell+2, 6\ell+3), (4\ell+3, 8\ell+5) \text{ and } (7\ell+3, 7\ell+4).$$

We now check these pairs to insure that they are a (B, k)-system. In the following table we list the numbers covered by the pairs:

Pairs (a_r, b_r)	**Range of** a_r	**Range of** b_r
$(i, 4\ell-i+2)$	$1, 2, \ldots, 2\ell$	$2\ell+2, 2\ell+3, \ldots, 4\ell+1$
$(4\ell+i+3, 8\ell-i+4)$	$4\ell+4, 4\ell+5, \ldots, 5\ell+2$	$7\ell+5, 7\ell+6, \ldots, 8\ell+3$
$(5\ell+i+2, 7\ell-i+3)$	$5\ell+3, 5\ell+4, \ldots, 6\ell+1$	$6\ell+4, 6\ell+5, \ldots, 7\ell+2$
$(2\ell+1, 6\ell+2)$	$2\ell+1$	$6\ell+2$
$(4\ell+2, 6\ell+3)$	$4\ell+2$	$6\ell+3$
$(4\ell+3, 8\ell+5)$	$4\ell+3$	$8\ell+5$
$(7\ell+3, 7\ell+4)$	$7\ell+3$	$7\ell+4$

Lemma T.3 (continued 3)

Proof (continued)....

$$\begin{array}{rl} (i, 4\ell-i+2) & \text{for } i \in \{1,2,\ldots,2\ell\}, \\ (4\ell+i+3, 8\ell-i+4) & \text{for } i \in \{1,2,\ldots,\ell-1\}, \\ (5\ell+i+2, 7\ell-i+3) & \text{for } i \in \{1,2,\ldots,\ell-1\}, \end{array}$$
$$(2\ell+1, 6\ell+2), (4\ell+2, 6\ell+3), (4\ell+3, 8\ell+5) \text{ and } (7\ell+3, 7\ell+4).$$

We now check these pairs to insure that they are a (B, k)-system. In the following table we list the values of $b_r - a_r$:

Pairs (a_r, b_r)	$b_r - a_r$	**Range of** $b_r - a_r$
$(i, 4\ell-i+2)$	$4\ell-2i+2$	$2, 4, \ldots, 4\ell$ even
$(4\ell+i+3, 8\ell-i+4)$	$4\ell-2i+1$	$2\ell+3, 2\ell+5, \ldots, 4\ell-1$ odd
$(5\ell+i+2, 7\ell-i+3)$	$2\ell-2i+1$	$3, 5, \ldots, 2\ell-1$ odd
$(2\ell+1, 6\ell+2)$	$4\ell+1$	$4\ell+1$
$(4\ell+2, 6\ell+3)$	$2\ell+1$	$2\ell+1$
$(4\ell+3, 8\ell+5)$	$4\ell+2$	$4\ell+2$
$(7\ell+3, 7\ell+4)$	1	1

Lemma T.3 (continued 4)

Proof (continued). Now suppose $k \equiv 3 \pmod 4$, say $k = 4\ell-1$. If $k = 3$ and $\ell = 1$, consider the pairs $(1,2)$, $(3,5)$, and $(4,7)$. If $k \geq 7$ (and so $\ell \geq 2$) then consider the pairs

$$\begin{array}{rl} (4\ell+i, 8\ell-i-2) & \text{for } i \in \{1,2,\ldots,2\ell-2\}, \\ (i, 4\ell-i-1) & \text{for } i \in \{1,2,\ldots,\ell-1\}, \\ (\ell+i+1, 3\ell-i) & \text{for } i \in \{1,2,\ldots,\ell-2\}, \end{array}$$
$$(\ell, \ell+1), (2\ell, 4\ell-1), (2\ell+1, 6\ell-1), \text{ and } (4\ell, 8\ell-1).$$

We now check these pairs to insure that they are a (B, k)-system. In the following table we list the numbers covered by the pairs:

Pairs (a_r, b_r)	**Range of** a_r	**Range of** b_r
$(4\ell+i, 8\ell-i-2)$	$4\ell+1, 4\ell+2, \ldots, 6\ell-2$	$6\ell, 6\ell+1, \ldots, 8\ell-3$
$(i, 4\ell-i-1)$	$1, 2, \ldots, \ell-1$	$3\ell, 3\ell+1, \ldots, 4\ell-2$
$(\ell+i+1, 3\ell-i)$	$\ell+2, \ell+3, \ldots, 2\ell-1$	$2\ell+2, 2\ell+3, \ldots, 3\ell-1$
$(\ell, \ell+1)$	ℓ	$\ell+1$
$(2\ell, 4\ell-1)$	2ℓ	$4\ell-1$
$(2\ell+1, 6\ell-1)$	$2\ell+1$	$6\ell-1$
$(4\ell, 8\ell-1)$	4ℓ	$8\ell-1$

Lemma T.3 (continued 5)

Proof (continued)....

$$(4\ell+i, 8\ell-i-2) \quad \text{for } i \in \{1,2,\ldots,2\ell-2\},$$
$$(i, 4\ell-i-1) \quad \text{for } i \in \{1,2,\ldots,\ell-1\},$$
$$(\ell+i+1, 3\ell-i) \quad \text{for } i \in \{1,2,\ldots,\ell-2\},$$
$$(\ell,\ell+1),(2\ell,4\ell-1),(2\ell+1,6\ell-1), \text{ and } (4\ell,8\ell-1).$$

We now check these pairs to insure that they are a (B, k)-system. In the following table we list the values of $b_r - a_r$:

Pairs (a_r, b_r)	$b_r - a_r$	**Range of** $b_r - a_r$
$(4\ell+i, 8\ell-i-2)$	$4\ell-2i-2$	$2,4,\ldots,4\ell-4$ even
$(i, 4\ell-i-1)$	$4\ell-2i-1$	$2\ell+1, 2\ell+3,\ldots,4\ell-3$ odd
$(\ell+i+1, 3\ell-i)$	$2\ell-2i-1$	$3,5,\ldots,2\ell-3$ odd
$(\ell,\ell+1)$	1	1
$(2\ell,4\ell-1)$	$2\ell-1$	$2\ell-1$
$(2\ell+1,6\ell-1)$	$4\ell-2$	$4\ell-2$
$(4\ell,8\ell-1)$	$4\ell-1$	$4\ell-1$

□

Lemma T.4

Lemma T.4. There exists a cyclic $STS(n)$ for all $n \equiv 1 \pmod 6$.

Proof. Let $n \equiv 1 \pmod 6$, say $n = 6k+1$. Let the (a_r, b_r)'s be from a (A, k)-system when $k \equiv 0$ or $1 \pmod 4$ and from a (B, k)-system when $k \equiv 2$ or $3 \pmod 4$. The triples

$$\{[j, r+j, b_r+k+j] \mid r \in \{1,2,\ldots,k\}, j \in \{0,1,\ldots,n-1\}\}$$

form a $STS(n)$. This is because a (A, k)-system and a (B, k)-system give solutions to Heffter's First Difference Problem and hence can be used to give the "base blocks" for cyclic $STS(n)$. □

Lemma T.6

Lemma T.6. There exists a cyclic $STS(n)$ for all $n \equiv 3 \pmod 6$, $n \neq 9$.

Proof. Let $n \equiv 3 \pmod 6$, say $n = 6k+3$ where $n \geq 15$. Let the (a_r, b_r)'s be from a (C, k)-system when $k \equiv 0$ or $1 \pmod 4$ and from a (D, k)-system when $k \equiv 2$ or $3 \pmod 4$. The triples

$$\{[j, r+j, b_r+k+j] \mid r \in \{1,2,\ldots,k\}, j \in \{0,1,\ldots,n/3-1\}\}$$

$$\cup\{[j, 2k+1+j, 4k+2+j] \mid j = 0,1,\ldots,n-1\}\}$$

form a $STS(n)$. This is because a (C, k)-system and a (D, k)-system give solutions to Heffter's Second Difference Problem and hence can be used to give the "base blocks" for cyclic $STS(n)$. □

Theorem T.1

Theorem T.1. A $STS(n)$ exists if and only if $n \equiv 1$ or $3 \pmod 6$.

Proof. The necessary conditions are given in Lemma T.1. A $STS(n)$, where $n \equiv 1 \pmod 6$, is shown to exist in Lemma T.4. A $STS(n)$, where $n \equiv 3 \pmod 6$, $n \neq 9$, is shown to exist in Lemma T.6. For $n = 9$, a $STS(9)$ is given by the triples $\{[0,1,2],[4,3,0],[2,8,4],[0,5,6],$ $[4,7,5],[2,6,7],[0,7,8],[4,6,1],[2,5,6],[1,3,7],[3,8,6],[8,1,5]\}$. Therefore, a $STS(n)$ exists if and only if $n \equiv 1$ or $3 \pmod 6$. □