Graph Theory

Chapter 1. Graphs
1.3. Automorphisms—Proofs of Theorems

Lemma 1.3.1

Lemma 1.3.1. If $x$ is a vertex of the graph $X$ and $g$ is an automorphism of $X$, then the vertex $y = x^g$ has the same valency as $x$. (That is, an automorphism preserves degrees of vertices.)

Proof. Let $N(x)$ be the set of vertices adjacent to $x$ ($N(x)$ is called the open neighborhood of $x$):

$$N(x) = \{y \in V \mid y \sim x\}.$$ 

Then $|N(x)|$ is the valency of $x$. Since $g$ is an automorphism then (by the definition of isomorphism) $x \sim y$ if and only if $x^g \sim y^g$. Now the set of vertices adjacent to $x^g$ is:

$$N(x^g) = \{z \in V \mid z \sim x^g\}.$$ 

So $z \in N(x^g)$ if and only if $z = y^g$ for some $y \in N(x)$. That is, $g : N(x) \rightarrow N(x^g)$. Since $g$ is a bijection, then $|N(x)| = |N(x^g)|$. That is, the valency of $x$ is the same as the valency of $x^g$, as claimed. 

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