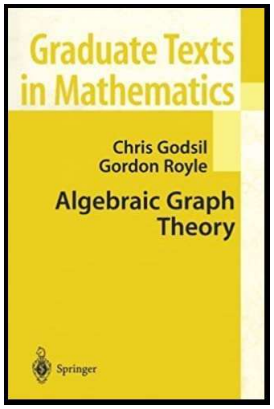
Graph Theory

Chapter 1. Graphs

1.3. Automorphisms—Proofs of Theorems



Lemma 1.3.1

Lemma 1.3.1. If x is a vertex of the graph X and g is an automorphism of X, then the vertex $y = x^g$ has the same valency as x. (That is, an automorphism preserves degrees of vertices.)

Proof. Let $N(x)$ be the set of vertices adjacent to x ($N(x)$ is called the *open neighborhood* of x):

$$N(x) = \{y \in V \mid y \sim x\}.$$

Then $|N(x)|$ is the valency of x. Since g is an automorphism then (by the definition of isomorphism) $x \sim y$ if and only if $x^g \sim y^g$. Now the set of vertices adjacent to x^g is:

$$N(x^g) = \{z \in V \mid z \sim x^g\}.$$

So $z \in N(x^g)$ if and only if $z = y^g$ for some $y \in N(x)$. That is, $g : N(x) \mapsto N(x^g)$. Since g is a bijection, then $|N(x)| = N(x^g)|$. That is, the valency of x is the same as the valency of x^g, as claimed. □