Graph Theory

Chapter 1. Graphs

1.4. Homomorphisms—Proofs of Theorems

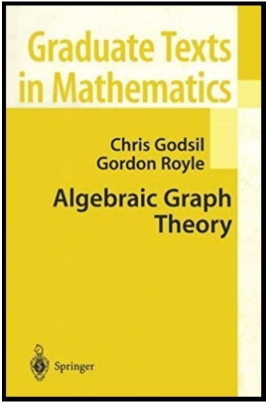

Table of contents

Lemma 1.4.1

Lemma 1.4.1. The chromatic number of a graph X is the least integer r such that there is a homomorphism from X to K_r.

Proof. Suppose f is any homomorphism from the graph X to the graph Y. If $y \in V(Y)$, then we have the inverse image $f^{-1}(\{y\}) = \{x \in V(X) \mid f(x) = y\}$. Since y is not adjacent to itself (because we only consider simple graphs) then $f^{-1}(\{y\})$ is an independent set (that is, no two elements of $f^{-1}(\{y\})$ are adjacent).

Lemma 1.4.1

Lemma 1.4.1. The chromatic number of a graph X is the least integer r such that there is a homomorphism from X to K_r.

Proof. Suppose f is any homomorphism from the graph X to the graph Y. If $y \in V(Y)$, then we have the inverse image $f^{-1}(\{y\}) = \{x \in V(X) \mid f(x) = y\}$. Since y is not adjacent to itself (because we only consider simple graphs) then $f^{-1}(\{y\})$ is an independent set (that is, no two elements of $f^{-1}(\{y\})$ are adjacent).

If there is a homomorphism from X to a graph with r vertices, say with vertex set $\{y_1, y_2, \ldots, y_r\}$, then the r sets $f^{-1}(\{y_i\})$, where $1 \leq i \leq r$, can each be given a unique colour (that is, the $f^{-1}(\{y_i\})$ form colour classes) resulting in a proper r-colouring of X. Hence $\chi(X) \leq r$; that is, if there is a homomorphism from X to a graph with r vertices then it is necessary that $\chi(X) \leq r$. In particular, if there is a homomorphism from X to K_r then we must have $\chi(X) \leq r$. To complete the proof we need to show that there is a homomorphism from X to $K_{\chi(X)}$.

Lemma 1.4.1

Lemma 1.4.1. The chromatic number of a graph X is the least integer r such that there is a homomorphism from X to K_r.

Proof. Suppose f is any homomorphism from the graph X to the graph Y. If $y \in V(Y)$, then we have the inverse image $f^{-1}(\{y\}) = \{x \in V(X) \mid f(x) = y\}$. Since y is not adjacent to itself (because we only consider simple graphs) then $f^{-1}(\{y\})$ is an independent set (that is, no two elements of $f^{-1}(\{y\})$ are adjacent).

If there is a homomorphism from X to a graph with r vertices, say with vertex set $\{y_1, y_2, \ldots, y_r\}$, then the r sets $f^{-1}(\{y_i\})$, where $1 \leq i \leq r$, can each be given a unique colour (that is, the $f^{-1}(\{y_i\})$ form colour classes) resulting in a proper r-colouring of X. Hence $\chi(X) \leq r$; that is, if there is a homomorphism from X to a graph with r vertices then it is necessary that $\chi(X) \leq r$. In particular, if there is a homomorphism from X to K_r then we must have $\chi(X) \leq r$. To complete the proof we need to show that there is a homomorphism from X to $K_{\chi(X)}$.

Lemma 1.4.1 (continued)

Lemma 1.4.1. The chromatic number of a graph X is the least integer r such that there is a homomorphism from X to K_r.

Proof (continued). Suppose that X can be properly coloured with r colours $\{1, 2, \ldots, r\}$. Let Y be the complete graph K_r where Y is given a proper colouring with colours $1, 2, \ldots, r$. Consider mapping f that sends each vertex of X to a vertex of $Y = K_r$ of the same colour (so f maps colour classes of f to vertices of Y of the same colour). Then $f(x)$ and $f(y)$ are adjacent in Y whenever x and y are adjacent in X (since every $f(x)$ is adjacent to every $f(y)$ because $Y = K_r$). That is, f is a homomorphism. In particular, with $r = \chi(X)$, we get a homomorphism from X to $K_r = K_{\chi(X)}$. By the argument above, we know that there is no homomorphism from X to K_s where $s < r$, so the least integer r such there is a homomorphism form X to K_r is the chromatic number $\chi(X)$, as claimed. □

Lemma 1.4.1 (continued)

Lemma 1.4.1. The chromatic number of a graph X is the least integer r such that there is a homomorphism from X to K_r.

Proof (continued). Suppose that X can be properly coloured with r colours $\{1, 2, \ldots, r\}$. Let Y be the complete graph K_r where Y is given a proper colouring with colours $1, 2, \ldots, r$. Consider mapping f that sends each vertex of X to a vertex of $Y = K_r$ of the same colour (so f maps colour classes of f to vertices of Y of the same colour). Then $f(x)$ and $f(y)$ are adjacent in Y whenever x and y are adjacent in X (since every $f(x)$ is adjacent to every $f(y)$ because $Y = K_r$). That is, f is a homomorphism. In particular, with $r = \chi(X)$, we get a homomorphism from X to $K_r = K_{\chi(X)}$. By the argument above, we know that there is no homomorphism from X to K_s where $s < r$, so the least integer r such there is a homomorphism form X to K_r is the chromatic number $\chi(X)$, as claimed. □