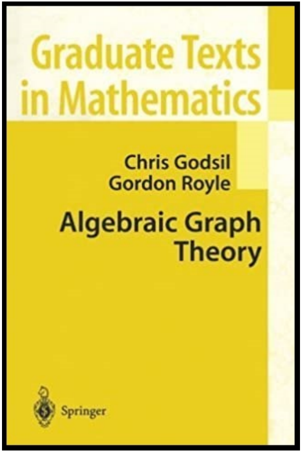
Graph Theory

Chapter 1. Graphs

1.6. Johnson Graphs—Proofs of Theorems



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Proof. Let g be the bijection mapping subsets of Ω of size k to subsets of Ω of size $v-k$ defined as $g(X) = \Omega \setminus X = X^c$ (so that g maps a k-set to its complement in Ω). If $X_1 \sim X_2$ in $J(v,k,i)$ then $|X_1 \cap X_2| = i$ (by definition). So

$$\begin{aligned}(\Omega \setminus X_1) \cap (\Omega \setminus X_2) &= X_1^c \cap X_2^c = (X_1 \cup X_2)^c \text{ by de Morgan's Law} \\ &= \Omega \setminus (X_1 \cup X_2),\end{aligned}$$

and so $|X_1^c \cap X_2^c| = |(\Omega \setminus X_1) \cap (\Omega \setminus X_2)| = |\Omega \setminus (X_1 \cup X_2)|$ $= |\Omega| - (k + k - i) = v - 2k + i$. Therefore, $X_1^c \sim X_2^c$ in $J(v, v-k, x-2k+i)$.

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Lemma 1.6.2

Lemma 1.6.2. If $v \geq k \geq i$, then $\text{Aut}(J(v,k,i))$ contains a subgroup isomorphic to $\text{Sym}(v)$.

Proof. Let g be any permutation of Ω (i.e, $g \in \text{Sym}(v)$). Then g is a bijection on Ω and so $|S \cap T| = |S^g \cap T^g|$ for all $S, T \subseteq \Omega$. Therefore $X_1 \sim X_2$ in $J(v,k,i)$ if and only if $X_1^g \sim X_2^g$ in $J(v,k,i)$ and so g is an automorphism of $J(v,k,i)$. Since g is an arbitrary permutation of Ω (where $|\Omega| = v$) then the group of all permutations of Ω (a group isomorphic to the symmetry group $\text{Sym}(v)$) are in $\text{Aut}(J(v,k,i))$. That is, $\text{Aut}(J(v,k,i))$ contains a subgroup isomorphic to Sym(v), as claimed. □

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