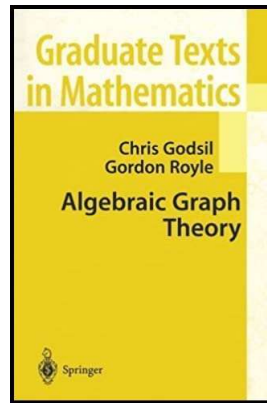


# Graph Theory

## Chapter 1. Graphs

### 1.7. Line Graphs—Proofs of Theorems



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Lemma 1.7.1

## Lemma 1.7.1

**Lemma 1.7.1.** If  $X$  is regular with valency (i.e., degree)  $k$ , then  $L(X)$  is regular with valency  $2k - 2$ .

**Proof.** Since each vertex of  $X$  is of valency  $k$ , then every edge of  $X$  is incident with  $k - 1$  edges at each of its ends. Hence, in  $L(X)$  each vertex of valency  $2(k - 1) = 2k - 2$ , as claimed.  $\square$

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Theorem 1.7.2

## Theorem 1.7.2

**Theorem 1.7.2.** A nonempty graph is a line graph if and only if its edge set can be partitioned into a set of cliques with the property that any vertex lies in at most two cliques.

**Proof.** Suppose nonempty graph  $G$  is the line graph of some graph  $X$ , so that  $G = L(X)$ . For any edge of  $G$ , there is a corresponding vertex  $x$  of  $X$ ; say the valency of  $x$  is  $k$ . Then in  $X$  there are  $k$  edges containing  $x$  and these edges are all incident with each other, so that the corresponding  $k$  vertices of  $G = L(X)$  are all adjacent to each other (in  $G$ ). That is, these vertices form a clique of  $G$  of size  $k$  (where  $k$  is equal to the valency of  $x$ ). So with every vertex of  $X$ , there is a corresponding clique in  $G = L(X)$  and hence every edge of  $G = L(X)$  lies in at least one clique of  $G$  (namely, the clique determined by the vertex corresponding to that edge); when  $X$  has  $n$  vertices,  $G = L(X)$  has  $n$  cliques.

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Theorem 1.7.2

## Theorem 1.7.2 (continued)

**Theorem 1.7.2.** A nonempty graph is a line graph if and only if its edge set can be partitioned into a set of cliques with the property that any vertex lies in at most two cliques.

**Proof (continued).** When we sum of the valencies of vertices in  $X$  we get the number of edges of  $G = L(X)$  (by the definition of adjacency in the line graph). Since this sum also equals the sum of the sizes of the cliques of  $G$ , as observed above, then each edge of  $G = L(X)$  lies in at most one of the cliques. Since we have that each edge of  $G = L(X)$  lies in at least one of the cliques, we now have that each edge of  $G = L(X)$  lies in exactly one of these cliques, as claimed.  $\square$

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## Theorem 1.7.3

**Lemma 1.7.3.** Suppose that  $X$  and  $Y$  are graphs with minimum valency four. Then  $X \cong Y$  if and only if  $L(X) \cong L(Y)$ .

**Proof.** If  $X \cong Y$ , the “clearly”  $L(X) \cong L(Y)$ , as observed above. Now, suppose  $L(X) \cong L(Y)$ . Let  $C$  be a clique in  $L(X)$  containing exactly  $c$  vertices. If  $c > 3$ , then the vertices of  $C$  correspond to a set of  $c$  edges in  $X$ , each incident in  $X$  to a common vertex. Then there is a bijection between the vertices of  $X$  and the maximal cliques of  $L(X)$  that takes adjacent vertices to pairs of cliques with a vertex in common (by the “any vertex lies in at most two cliques” of Theorem 1.7.2). The remaining details that  $L(X)$  and  $L(Y)$  have the same adjacency (up to isomorphism) are left as an exercise.  $\square$

## Theorem 1.7.5

**Lemma 1.7.5.** If the line graph of a connected graph  $X$  is regular, then  $X$  is regular or bipartite and semiregular.

**Proof.** Suppose  $X$  is connected and that  $L(X)$  is regular with valency  $k$ . If  $u$  and  $v$  are adjacent vertices in  $X$ , then their valencies sum to  $k + 2$  (because in this sum the edge  $uv$  is counted twice, so that the vertex of  $L(X)$  corresponding to edge  $uv$  of  $X$  is of valency  $k$ ). Since  $X$  is connected, this implies that there are at most two different valencies, namely  $\lfloor k + 2 \rfloor$  and  $\lceil k + 2 \rceil$ . So  $X$  is either regular or semiregular.

## Theorem 1.7.5 (continued)

**Lemma 1.7.5.** If the line graph of a connected graph  $X$  is regular, then  $X$  is regular or bipartite and semiregular.

**Proof (continued).** If two adjacent vertices have the same valency, then it can be shown by induction that  $X$  is regular. If  $X$  contains a cycle of odd length, then it must have two adjacent vertices of the same valency (just as the chromatic number of an odd cycle is three), and so by the previous observation  $X$  is regular. Therefore, if  $X$  is not regular, then it has no cycle of odd length. Such a graph is bipartite; see my online notes for Graph Theory 1 (MATH 5340) on Section 4.2. Spanning Trees (notice Theorem 4.7). Therefore,  $X$  is either regular or bipartite and semiregular, as claimed.  $\square$