Graph Theory

Chapter 1. Graphs

1.7. Line Graphs—Proofs of Theorems

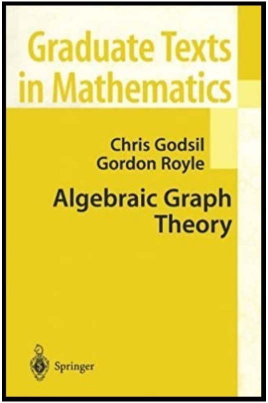

Table of contents

Lemma 1.7.1

Lemma 1.7.1. If X is regular with valency (i.e., degree) k, then $L(X)$ is regular with valency $2k - 2$.

Proof. Since each vertex of of X is of valency k, then every edge of X is incident with $k - 1$ edges at each of its ends. Hence, in $L(X)$ each vertex of of valency $2(k - 1) = 2k - 2$, as claimed. □

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Theorem 1.7.2

Theorem 1.7.2. A nonempty graph is a line graph if and only if its edge set can be partitioned into a set of cliques with the property that any vertex lies in at most two cliques.

Proof. Suppose nonempty graph G is the line graph of some graph X, so that $G = L(X)$. For any edge of G, there is a corresponding vertex x of X; say the valency of x is k. Then in X there are k edges containing x and these edges are all incident with each other, so that the corresponding k vertices of $G = L(X)$ are all adjacent to each other (in G). That is, these vertices form a clique of G of size k (where k is equal to the valency of x).

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Theorem 1.7.2 (continued)

Theorem 1.7.2. A nonempty graph is a line graph if and only if its edge set can be partitioned into a set of cliques with the property that any vertex lies in at most two cliques.

Proof (continued). When we sum of the valencies of vertices in X we get the number of edges of $G = L(X)$ (by the definition of adjacency in the line graph). Since this sum also equals the sum of the sizes of the cliques of G, as observed above, then each edge of $G = L(X)$ lies in at most one of the cliques. Since we have that each edge of $G = L(X)$ lies in at least one of the cliques, we now have that each edge of $G = L(X)$ lies in exactly one of these cliques, as claimed. □

Theorem 1.7.3

Lemma 1.7.3. Suppose that X and Y are graphs with minimum valency four. Then $X \cong Y$ if and only if $L(X) \cong L(Y)$.

Proof. If $X \cong Y$, the "clearly" $L(X) \cong L(Y)$, as observed above. Now, suppose $L(X) \cong L(Y)$. Let C be a clique in $L(X)$ containing exactly c vertices. If $c > 3$, then the vertices of C correspond to a set of c edges in X, each incident in X to a common vertex. Then there is a bijection between the vertices of X and the maximal cliques of $L(X)$ that takes adjacent vertices to pairs of cliques with a vertex in common (by the "any vertex lies in at most two cliques" of Theorem 1.7.2). The remaining details that $L(X)$ and $L(Y)$ have the same adjacency (up to isomorphism) are left as and exercise. □

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Theorem 1.7.5

Lemma 1.7.5. If the line graph of a connected graph X is regular, then X is regular or bipartite and semiregular.

Proof. Suppose X is connected and that $L(X)$ is regular with valency k. If u and v are adjacent vertices in X, then their valencies sum to $k+2$ (because in this sum the edge uv is counted twice, so that the vertex of $L(X)$ corresponding to edge uv of X is of valency k). Since X is connect, this implies that there are at most two different valencies, namely $\lfloor k+2 \rfloor$ and $\lceil k+2 \rceil$. So X is either regular or semiregular.

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Theorem 1.7.5 (continued)

Lemma 1.7.5. If the line graph of a connected graph X is regular, then X is regular or bipartite and semiregular.

Proof (continued). If two adjacent vertices have the same valency, then it can be shown by induction that X is regular. If X contains a cycle of odd length, then it must have two adjacent vertices of the same valency (just a the chromatic number of an odd cycle is three), and so by the previous observation X is regular. Therefore, if X is not regular, then it has not cycle of odd length. Such a graph if bipartite; see my online notes for Graph Theory 1 (MATH 5340) on Section 4.2. Spanning Trees (notice Theorem 4.7). Therefore, X is either regular or bipartite and semiregular, as claimed. $\square$