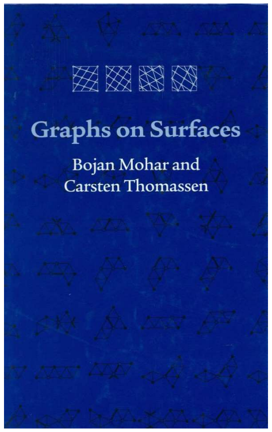
Graph Theory

Chapter 1. Introduction

1.2. Trees and Bipartite Graphs—Proofs of Theorems



Proposition 1.2.1

Proposition 1.2.1. Let T be a graph of order n. Then the following are equivalent.

(i) T is a tree.

(ii) T is connected and has $n-1$ edges.

(iii) T contains no cycles and has $n-1$ edges.

(iv) T is connected but every edge-deletion results in a disconnected graph.

(v) T contains no cycles but every edge addition results in a graph with a cycle.

(vi) Any two vertices in T are connected by exactly one path.

Proof. Bondy and Murty's Exercise 4.1.2 shows that (i), (ii), and (iii) are equivalent. Their Proposition 4.1 shows that (i) and (vi).

Proposition 1.2.1 (continued 1)

Proof (continued). (vi) $\Rightarrow$ (i). Suppose any two vertices in T are connected by exactly one path. Then by Mohar and Thomassen's definition of "connected," T is connected. ASSUME T contains a cycle $v_0, v_1, \ldots, v_{n-1}$ with v_1 and v_j adjacent if and only if either $i \equiv j \pmod{n}$ or $j \equiv i \pmod{n}$ (since T is simple, $n \geq 2$), then there are two paths between v_0 and v_1, namely edge v_0v_1 and path $v_1, v_2, \ldots, v_{n-1}, v_0$. But this is a CONTRADICTION to the hypothesis that T contains only one path between any two give vertices. So T contains no cycles and hence T is a tree.

(i) $\Rightarrow$ (v). Suppose T is a tree. Then, by the definition of "tree," T is connected. Let $u, v \in V(T)$ where $uv \notin E(T)$. Then condition (vi) holds and there is exactly one path between u and v. If we add edge uv then the path between u and v union with edge uv to give a cycle. Since u and v are arbitrary non-adjacent vertices in T, then addition of any edge to T results in a graph with a cycle, as claimed.

Proposition 1.2.1 (continued 2)

Proof (continued). (v) $\Rightarrow$ (i). Suppose T contains no cycles but that every edge addition results in a graph with a cycle. ASSUME T is not connected. Then by Mohar and Thomassen's definition of connected, there are $u, v \in V(T)$ such that there is not path between u and v in T. Then the addition of edge uv to T does not result in a graph with a cycle (for if edge uv is in some cycle C in the graph, then $C - uv$ is a path in T between vertices u and v). But this CONTRADICTS the hypotheses. So the assumption that T is not connected is false, and hence T is a connected graph with no cycles. That is, T is a tree, as claimed.

(i) $\Rightarrow$ (iv). Suppose T is a tree. Then, by the definition of "tree," T is connected. Let uv be an edge of T. Then condition (vi) holds and there is exactly one path between u and v and it must be the edge uv. So if edge uv is deleted from T then there is no path between u and v in the resulting graph and by Mohar and Thomassen's definition of "connected," the resulting graph is not connected. Since uv is an arbitrary edge of T then any deletion of an edge of T results in a disconnected graph. $\square$