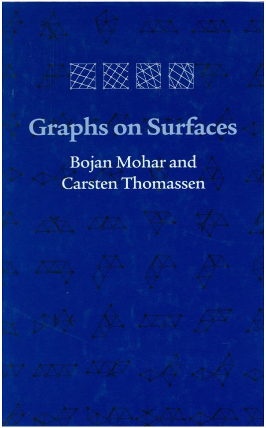
Graph Theory

Chapter 1. Introduction

1.2. Trees and Bipartite Graphs—Proofs of Theorems



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Proposition 1.2.1

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(i) T is a tree.

(ii) T is connected and has $n-1$ edges.

(iii) T contains no cycles and has $n-1$ edges.

(iv) T is connected but every edge-deletion results in a disconnected graph.

(v) T contains no cycles but every edge addition results in a graph with a cycle.

(vi) Any two vertices in T are connected by exactly one path.

Proof. Bondy and Murty's Exercise 4.1.2 shows that (i), (ii), and (iii) are equivalent. Their Proposition 4.1 shows that (i) and (vi).

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Proposition 1.2.1 (continued 1)

Proof (continued). (vi) $\Rightarrow$ (i). Suppose any two vertices in T are connected by exactly one path. Then by Mohar and Thomassen's definition of "connected," T is connected. ASSUME T contains a cycle $v_0, v_1, \ldots, v_{n-1}$ with v_1 and v_j adjacent if and only if either $i \equiv j \pmod{n}$ or $j \equiv i \pmod{n}$ (since T is simple, $n \geq 2$), then there are two paths between v_0 and v_1, namely edge $v_0 v_1$ and path $v_1, v_2, \ldots, v_{n-1}, v_0$. But this is a CONTRADICTION to the hypothesis that T contains only one path between any two give vertices. So T contains no cycles and hence T is a tree.

(i) $\Rightarrow$ (v). Suppose T is a tree. Then, by the definition of "tree," T is connected. Let $u, v \in V(T)$ where $uv \notin E(T)$. Then condition (vi) holds and there is exactly one path between u and v. If we add edge uv then the path between u and v union with edge uv to give a cycle. Since u and v are arbitrary non-adjacent vertices in T, then addition of any edge to T results in a graph with a cycle, as claimed.

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Proposition 1.2.1 (continued 2)

Proof (continued). (v) $\Rightarrow$ (i). Suppose T contains no cycles but that every edge addition results in a graph with a cycle. ASSUME T is not connected. Then by Mohar and Thomassen's definition of connected, there are $u, v \in V(T)$ such that there is not path between u and v in T. Then the addition of edge uv to T does not result in a graph with a cycle (for if edge uv is in some cycle C in the graph, then $C - uv$ is a path in T between vertices u and v). But this CONTRADICTS the hypotheses. So the assumption that T is not connected is false, and hence T is a connected graph with no cycles. That is, T is a tree, as claimed.

(i) $\Rightarrow$ (iv). Suppose T is a tree. Then, by the definition of "tree," T is connected. Let uv be an edge of T. Then condition (vi) holds and there is exactly one path between u and v and it must be the edge uv. So if edge uv is deleted from T then there is no path between u and v in the resulting graph and by Mohar and Thomassen's definition of "connected," the resulting graph is not connected. Since uv is an arbitrary edge of T then any deletion of an edge of T results in a disconnected graph. $\square$

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Proof (continued). (v) $\Rightarrow$ (i). Suppose T contains no cycles but that every edge addition results in a graph with a cycle. ASSUME T is not connected. Then by Mohar and Thomassen's definition of connected, there are $u, v \in V(T)$ such that there is not path between u and v in T. Then the addition of edge uv to T does not result in a graph with a cycle (for if edge uv is in some cycle C in the graph, then $C - uv$ is a path in T between vertices u and v). But this CONTRADICTS the hypotheses. So the assumption that T is not connected is false, and hence T is a connected graph with no cycles. That is, T is a tree, as claimed.

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