Graph Theory 1, MATH 5340, Fall 2020

Homework 9, Supplement. Graph Decompositions: Triple Systems

Due Sunday, November 1, at noon

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, class notes, or hypotheses. Do not copy the work of others; **do your own work!!!**

T.1. **(a)** Prove that a necessary condition for the existence of a (C, k)-system is that $k \equiv 0$ or $3 \pmod 4$.

(b) Suppose $k \equiv 0 \pmod 4$, say $k = 4\ell$. Consider the pairs:

$$\begin{array}{rl} (i, 4\ell - i + 1) & i \in \{1, 2, \ldots, \ell - 1\} \\ (\ell + i - 1, 3\ell - i) & i \in \{1, 2, \ldots, \ell - 1\} \\ (4\ell + i + 1, 8\ell - i + 1) & i \in \{1, 2, \ldots, \ell - 1\} \\ (5\ell + i + 1, 7\ell - i + 1) & i \in \{1, 2, \ldots, \ell - 1\} \end{array}$$
$$(2\ell - 1, 2\ell), (3\ell, 5\ell + 1), (3\ell + 1, 7\ell + 1), \text{ and } (6\ell + 1, 8\ell + 1).$$

As in the supplement "Graph Decompositions: Triple Systems," show that the range of a_r and b_r for all r is as required (namely, $\{1, 2, \ldots, k, k + 2, k + 3, \ldots, 2k + 1\}$) and that $b_r - a_r$ has the range as required (namely, $\{1, 2, \ldots, k\}$).

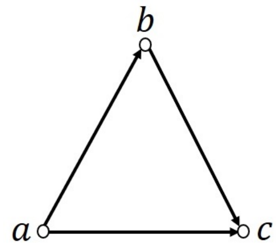
T.2. **(a)** Prove that a necessary condition for the existence of a (D, k)-system is that $k \equiv 1$ or $2 \pmod 4$, $k \neq 1$.

(b) Suppose $k \equiv 1 \pmod 4$, say $k = 4\ell + 1$. If $k = 5$ and $\ell = 1$, consider the pairs $(1, 5)$, $(2, 7)$, $(3, 4)$, $(8, 10)$, and $(9, 12)$. If $k \geq 9$ (and so $\ell \geq 2$) then consider the pairs:

$$\begin{array}{rl} (i, 4\ell - i + 2) & i \in \{1, 2, \ldots, 2\ell\} \\ (5\ell + i, 7\ell - i + 3) & i \in \{1, 2, \ldots, \ell\} \\ (4\ell + i + 2, 8\ell - i + 3) & i \in \{1, 2, \ldots, \ell - 2\} \end{array}$$
$$(2\ell + 1, 6\ell + 2), (6\ell + 1, 8\ell + 4), \text{ and } (7\ell + 3, 7\ell + 4).$$

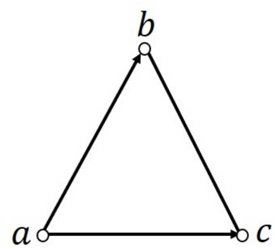
As in the supplement "Graph Decompositions: Triple Systems," show that the range of a_r and b_r for all r is as required (namely, $\{1, 2, \ldots, k, k+2, k+3, \ldots, 2k, 2k+2\}$) and that $b_r - a_r$ has the range as required (namely, $\{1, 2, \ldots, k\}$).

T.3. Let D_n be a complete digraph with vertex set $V(D_n) = \{0, 1, 2, \ldots, n-1\}$. With arc $(x, y) \in A(D_n)$ we associate the *difference* $(y - x) \pmod{n}$. Notice that if the differences associated with a directed triple are d_1, d_2, and d_3 then we must have $d_1 + d_2 \equiv d_3 \pmod{n}$. In 1982, Charlie and Marlene Colbourn proved that a cyclic $DTS(n)$ exists if and only if $n \equiv 1$, 4, or 7 (mod 12) (in "The Analysis of Directed Triple Systems by Refinement," *North-Holland Mathematics Studies*, **65** (1982), 97–103). Use an (A, k)-system to prove that a cyclic $DTS(n)$ exists for $n \equiv 1$ or 4 (mod 12).



Directed Triple

T.4. In my paper "Triple Systems from Mixed Graphs" (*Bulletin of the Institute of Combinatorics and its Applications*, **27** (1999) 95–100), it is shown that a T_1-triple system of order n exists if and only if $n \equiv 1 \pmod{2}$.



T_1

(a) Prove that the condition $n \equiv 1 \pmod{2}$ is a necessary condition for the existence of a T_1-triple system.

(b) For the case $n \equiv 1 \pmod{4}$, say $n = 4\ell + 1$, the collection of T_1 triples

$$[j, 2\ell - i + j, 2\ell + 1 + i + j]_1 \text{ for } i \in \{0, 1, \ldots, \ell - 1\} \text{ and } j \in \{0, 1, \ldots, 4\ell\}$$

$$[j, 4\ell - i + j, 1 + i + j]_1 \text{ for } i \in \{0, 1, \ldots, \ell - 1\} \text{ and } j \in \{0, 1, \ldots, 4\ell\}$$

are claimed to show the existence of a T_1-triple system. Explain why this is the case.