Graph Theory 1, MATH 5340, Fall 2024

Homework 3, 1.2. Isomorphisms and Automorphisms, 1.3. Graphs Arising from Other Structures

Due Saturday, September 14, at 11:59 p.m.

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, class notes, or hypotheses. Do not discuss homework problems with others. If you have any questions, then contact me (gardnerr@etsu.edu). Use the same notation and terminology we used in class and given in the in-class hints.

1.2.10. (a1) Show that, for $n \geq 2$, $\mathrm{Aut}(P_n) \cong S_2$.

1.3.3. Show that the line graph of $K_{3,3}$ is self-complementary.

1.3.A. CAYLEY DIGRAPH.Let G be a group and let C be an arbitrary subset of G, then the *Cayley digraph* $X(G, C)$ is the digraph with vertex set G and arc set $\{(g, h) \mid hg^{-1} \in C\}$. This definition is from Chris Godsil and Gordon Royle's *Algebraic Graph Theory*, Graduate Texts in Mathematics 207 (Springer, 2001); see page 35 or my online notes from this source on Section 3.1. Vertex-Transitive Graphs. John Fraleigh in *A First Course in Abstract Algebra*, 7th edition (Pearson, 2003) deals with this differently by having arc (g, h) in the Cayley digraph when $g^{-1}h \in C$ (see Fraleigh's page 70; the same approach appears in Fraleigh's 8th edition [Pearson, 2021] on page 72). It is "standard practice" in this setting to replace two arcs of the form (g, h) and (h, g) with the edge $\{g, h\}$. The dihedral group D_5 is isomorphic to the automorphism group of the 5-cycle, $D_5 \cong \mathrm{Aut}(C_5)$. This group is generated by two elements, ρ and μ, which satisfy $\mu^2 = e$, $\rho^5 = e$, and $\rho\mu = \mu\rho^{-1}$ (where e is the identity of D_5). The Cayley table of D_5 is given below.

$\times$	e	ρ	ρ^2	ρ^3	ρ^4	μ	$\mu\rho$	$\mu\rho^2$	$\mu\rho^3$	$\mu\rho^4$
e	e	ρ	ρ^2	ρ^3	ρ^4	μ	$\mu\rho$	$\mu\rho^2$	$\mu\rho^3$	$\mu\rho^4$
ρ	ρ	ρ^2	ρ^3	ρ^4	e	$\mu\rho^4$	μ	$\mu\rho$	$\mu\rho^2$	$\mu\rho^3$
ρ^2	ρ^2	ρ^3	ρ^4	e	ρ	$\mu\rho^3$	$\mu\rho^4$	μ	$\mu\rho$	$\mu\rho^2$
ρ^3	ρ^3	ρ^4	e	ρ	ρ^2	$\mu\rho^2$	$\mu\rho^3$	$\mu\rho^4$	μ	$\mu\rho$
ρ^4	ρ^4	e	ρ	ρ^2	ρ^3	$\mu\rho$	$\mu\rho^2$	$\mu\rho^3$	$\mu\rho^4$	μ
μ	μ	$\mu\rho$	$\mu\rho^2$	$\mu\rho^3$	$\mu\rho^4$	e	ρ	ρ^2	ρ^3	ρ^4
$\mu\rho$	$\mu\rho$	$\mu\rho^2$	$\mu\rho^3$	$\mu\rho^4$	μ	ρ^4	e	ρ	ρ^2	ρ^3
$\mu\rho^2$	$\mu\rho^2$	$\mu\rho^3$	$\mu\rho^4$	μ	$\mu\rho$	ρ^3	ρ^4	e	ρ	ρ^2
$\mu\rho^3$	$\mu\rho^3$	$\mu\rho^4$	μ	$\mu\rho$	$\mu\rho^2$	ρ^2	ρ^3	ρ^4	e	ρ
$\mu\rho^4$	$\mu\rho^4$	μ	$\mu\rho$	$\mu\rho^2$	$\mu\rho^3$	ρ	ρ^2	ρ^3	ρ^4	e

Find the Cayley graph of $X(D_5, C)$ where $C = \{\rho, \mu\}$, and use the "standard practice" concerning edges.