Graph Theory 2, MATH 5450, Spring 2021

Homework 7, 13.3. Variance

Due Friday, March 26, at 1:40

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook, class notes, or hypotheses. Do not copy the work of others; **do your own work!!!**

13.3.1. Let $X = \sum_{i=1}^n X_i$ be a sum of random variables on a finite probability space.

(a) Prove that

$$V(X) = \sum_{i=1}^{n} V(X_i) + \sum_{i \neq j} C(X_i, X_j).$$

(b) Deduce that if the X_i are indicator random variables, then

$$V(X) \leq E(X) + \sum_{i \neq j} C(X_i, X_j).$$

13.3.4. Chernoff's Inequality.

Let X_i, for $1 \leq i \leq n$, be independent random variables such that $P(X_i = +1) = P(X_i = -1) = 1/2$ for $1 \leq i \leq n$, and let $X = \sum_{i=1}^n X_i$.

(b) BONUS. Deduce that $E(e^{\alpha X}) \leq e^{\alpha^2 n/2}$. HINT: You may assume part (a) in which it is established that the random variables $e^{\alpha X_i}$, for $1 \leq i \leq n$, are independent, and $E(e^{\alpha X_i}) \leq e^{\alpha^2/2}$ for $1 \leq i \leq n$. WARNING: Do not simply assume that the expectation of a product of independent random variables is the product of the expectations (though this is true), but instead show computations for these particular random variables!

(c) By applying Markov's Inequality and choosing an appropriate value of α, derive the following concentration bound, valid for all $t > 0$, known as *Chernoff's Inequality*:

$$P(X \geq t) \leq e^{-t^2/(2n)}.$$