Section 1.2. Isomorphisms and Automorphisms

Note. In this section we define a graph isomorphism, consider automorphisms and symmetries of a given graph, and define a labeled simple graph.

Definition. Graphs G and H are *isomorphic*, denoted $G \cong H$, if there are bijections $\theta : V(G) \to V(H)$ and $\varphi : E(G) \to E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\varphi(e)) = \theta(u)\theta(v)$. The mappings θ and φ are an *isomorphism* between G and H.

Note. Figure 1.6 gives drawings of graphs G and H (with two drawings of H). Graphs G and H are isomorphic where (in a notation of a mapping similar to that given in Introduction to Modern Algebra [MATH 4127/5127] when dealing with permutations, see my online notes on II.8. Groups of Permutations):

$$\theta = \begin{pmatrix} a & b & c & d \\ w & z & y & x \end{pmatrix} \text{ and } \varphi = \begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ f_3 & f_4 & f_1 & f_6 & f_5 & f_2 \end{pmatrix}.$$

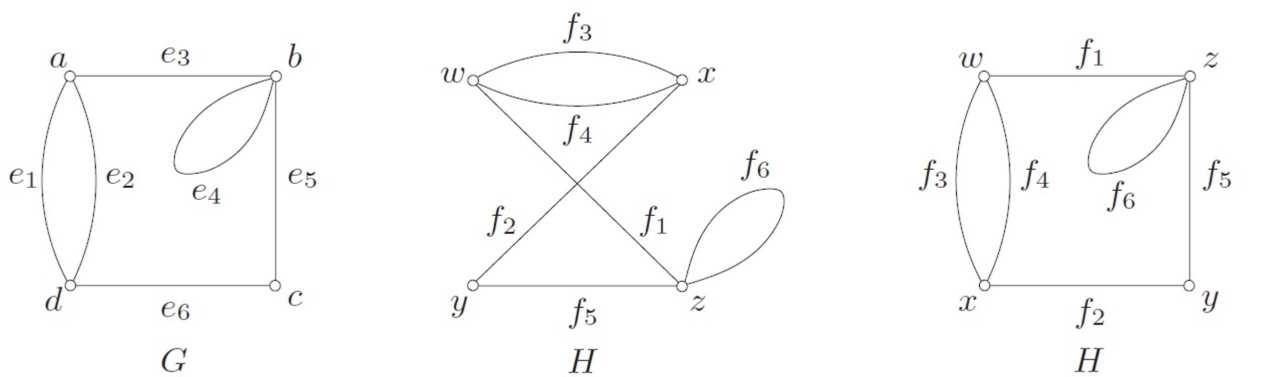

Figure 1.6

Note. Since θ and φ are bijections, then isomorphic graphs have the same number of vertices and edges (i.e., have the same order and size). Since two drawings of isomorphic graphs could be translated from one to the other (by renaming the edges and vertices) we often present an unlabeled drawing with the understanding that it represents an equivalence class of isomorphic graphs (since graph isomorphism is an equivalence relation).

Note. A complete graph on n vertices is denoted K_n. A complete bipartite graph, with partite sets consisting of m elements and n elements, is denoted $K_{m,n}$. A *path* on n vertices is denoted P_n and a *cycle* on n vertices is denoted C_n.

Note. Technically, K_n, $K_{m,n}$, P_n, and C_n each denote equivalence classes of graphs (for given $m, n \in \mathbb{N}$), but we may informally refer to "the complete graph K_n" (for example) with the understanding that this represents all graphs isomorphic to some complete graph on n vertices.

Note. It can be difficult to determine if two graphs are isomorphic or not (in particular, if they are "large"); see Bondy and Murty's discussion on pages 14 and 15. E. M. Luks, "employing powerful group theoretic methods" gave an efficient isomorphism-testing algorithm for graphs of bounded maximum degree in "Isomorphisms of Graphs of Bounded Valence can be Tested in Polynomial Time," *Journal of Computational System Science* **25**(1) (1982), 42–65 (available online from the index of volume 25 issue 1 where there is a link to a downloadable PDF of the paper; accessed 2/20/2020).

Note. In a simple graph G where the incidence function ψ_G is one to one (this is shown as part of the solution to Exercise 1.1.1), an edge is uniquely determined by the unordered pair of distinct vertices which are its ends. So an isomorphism between simple graphs can be determined from the mapping of the vertex sets, θ. This is the case in Mohar and Thomassen's book *Graphs on Surfaces* (Johns Hopkins University Press, 2001) which we use as a supplement when considering topological graph theory. See their 1.1. Basic Definitions for their approach to isomorphisms.

Definition. An *automorphism* of a graph is an isomorphism of the graph with itself. For vertices u and v in a simple graph G, if there is an automorphism of G with $\theta : V(G) \to V(G)$, such that $\theta(u) = v$ then vertices u and v are called *similar*. Simple graphs in which all vertices are similar are *vertex-transitive* graphs. Graphs in which no two vertices are similar are *asymmetric* graphs (and the only automorphism is the identity mapping).

Note. The complete graph K_n is clearly vertex transitive (since every one of the $n!$ mappings of $V(K_n)$ onto $V(K_n)$ is an automorphism).

Note 1.2.A. Drawings can help illustrate symmetries of a graph. Three drawings of the *Petersen graph* are given in Figure 1.9. The first drawing reveals a "rotational symmetry" in the five vertices of the inner pentagon and the five vertices of the outer pentagon. The third drawing reveals a similarity of the outer six vertices (through a "reflection" or "rotation"). The combination of these observations imply that there are automorphisms mapping any vertex of the Petersen graph to any other vertex of the Petersen graph; that is, the Petersen graph is vertex-transitive (in the two pentagons the vertices are similar and in the six cycle the vertices are similar, so it must be that at least one of the vertices in the outer pentagon is similar to at least one of the vertices of the inner pentagon and hence all vertices are similar because similarity is an equivalence relation, which is to be shown in Exercise 1.2.A).

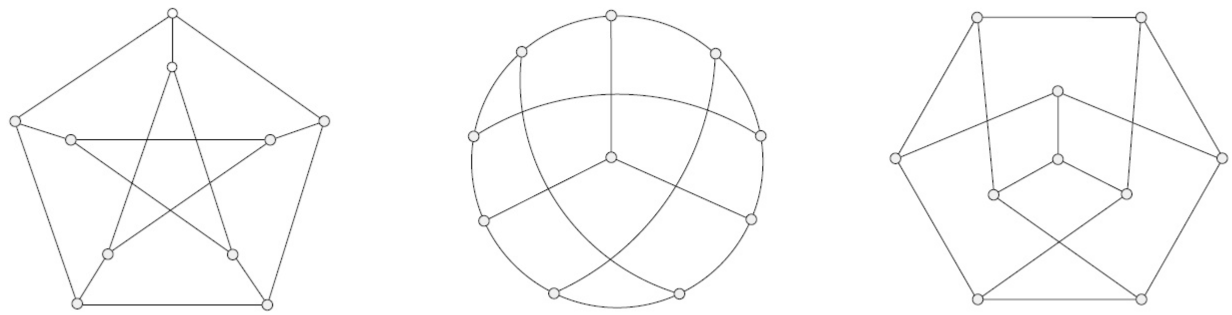

Figure 1.9

Definition. The set of all automorphisms of a graph G is denoted $\text{Aut}(G)$ and we denote $|\text{Aut}(G)|$ as $\text{aut}(G)$. It is to be shown in Exercise 1.2.9 that $\text{Aut}(G)$ is a group under function composition and is called the *automorphism group* of graph G.

Note. We have $\text{Aut}(K_n) = S_n$ where S_n is the symmetry group of all permutations on n elements. Any simple graph of order n will have an automorphism group which is a subgroup of S_n. In Exercise 1.2.10 it is to be shown that $\text{Aut}(C_n) = D_n$ where D_n is the dihedral group on n elements.

Definition. A simple graph whose vertices are labeled (usually with the labels $v_1, v_2, \ldots, v_n$) but whose edges are not labeled is a *labeled simple graph*.

Note 1.2.B. If a simple graph G has vertex set V where $|V| = n$, then there are at most $\binom{n}{2}$ possible edges of G. So

$$\text{the number of distinct labeled simple graphs of order } n \text{ is } 2^{\binom{n}{2}} \qquad (*)$$

(a particular possible edge may or may not be in $E(G)$, giving two choices for the presence/absence of an edge). We denote by $\mathcal{G}_n$ the set of labeled simple graphs with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The $2^{\binom{3}{2}} = 2^3 = 8$ different labeled simple graphs of order $n = 3$ are given in Figure 1.10.

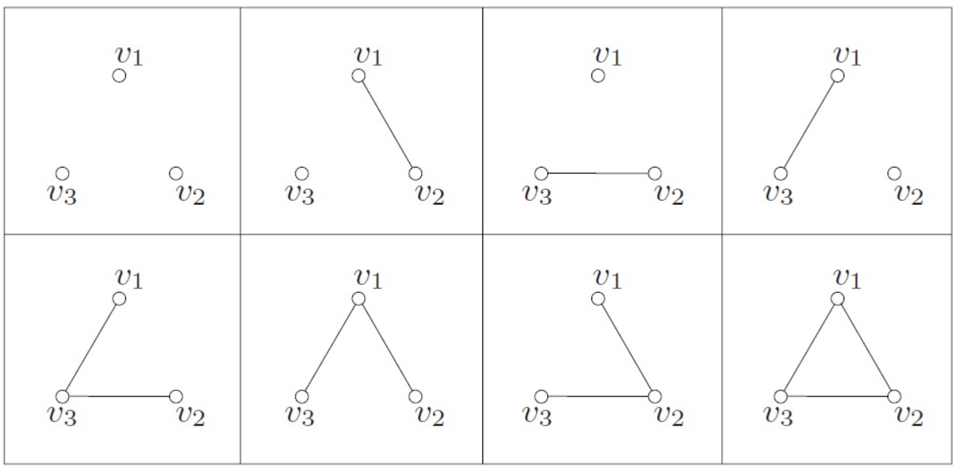

Figure 1.10

Note. Given n vertices, there are $n!$ ways to assign the labels $v_1, v_2, \ldots, v_n$. But for a given graph (with an assigned edge set), two of these labelings will yield the same labeled graphs if there is an automorphism of the graph mapping one labeling to the other. So

$$\text{the number of distinct labelings of a given unlabeled simple graph } G \text{ on } n \text{ vertices is } \frac{n!}{|\mathrm{Aut}(G)|} = \frac{n!}{\mathrm{aut}(G)}, \qquad (**)$$

as is to be argued in more detail in Exercise 1.2.15. For example, for K_3 there are $3! = 6$ labelings of the vertices, but $\mathrm{Aut}(K_3) = S_3$ and $\mathrm{aut}(K_3) = |\mathrm{Aut}(K_3)| = |S_3| = 3! = 6$ so that the number of distinct labelings of K_3 is $\dfrac{n!}{\mathrm{aut}(G)} = \dfrac{6}{6} = 1$. For P_3 there are $3! = 6$ labelings of the vertices, but $\mathrm{Aut}(P_3) = \mathbb{Z}_2$ (it consists of the identity and a reflection about its "center") and $\mathrm{aut}(P_3) = 2$ so that the number of distinct labelings of P_3 is $\dfrac{n!}{\mathrm{aut}(G)} = \dfrac{6}{2} = 3$; these three labelings are given as part of Figure 1.10.

Note. Combining $(*)$ and $(**)$, we have by summing over all unlabeled simple graphs G on n vertices that

$$\sum_G \frac{n!}{\mathrm{aut}(G)} = 2^{\binom{n}{2}}. \qquad (***)$$

For any such G, $\mathrm{aut}(G) \geq 1$ (since $\mathrm{Aut}(G)$ contains the identity) and so $\dfrac{1}{\mathrm{aut}(G)} \leq 1$. We therefore have from $(***)$ that

$$\frac{2^{\binom{n}{2}}}{n!} = \sum_G \frac{1}{\mathrm{aut}(G)} \leq \sum_G 1 = \left(\begin{array}{c} \text{the number of unlabeled} \\ \text{simple graphs } G \text{ of order } n \end{array} \right).$$

Since the number of unlabeled simple graphs of order n is a natural number, then we can round up on the left to get a lower bound on the number of unlabeled simple graphs of order n of $\left\lceil \dfrac{2^{\binom{n}{2}}}{n!} \right\rceil$. Bondy and Murty (see page 17) observe that this bound may not be good when n is small, but that when n is large the bound is a good approximation because the "vast majority" of graphs are asymmetric (that is, the proportion of simple graphs on n vertices that are asymmetric tends to 1 as n tends to infinity; see Exercise 1.2.15(d)).