Section 1.3. Graphs Arising from Other Structures

Note. In this section we define hypergraphs, geometric configurations, and particular graphs constructed from these.

Definition. A *polyhedral graph* is a graph whose vertices and edges are just the vertices and edges of the polyhedron with the same incidence relation.

Note. The five platonic solids give rise to the five "platonic graphs" of Figure 1.14.

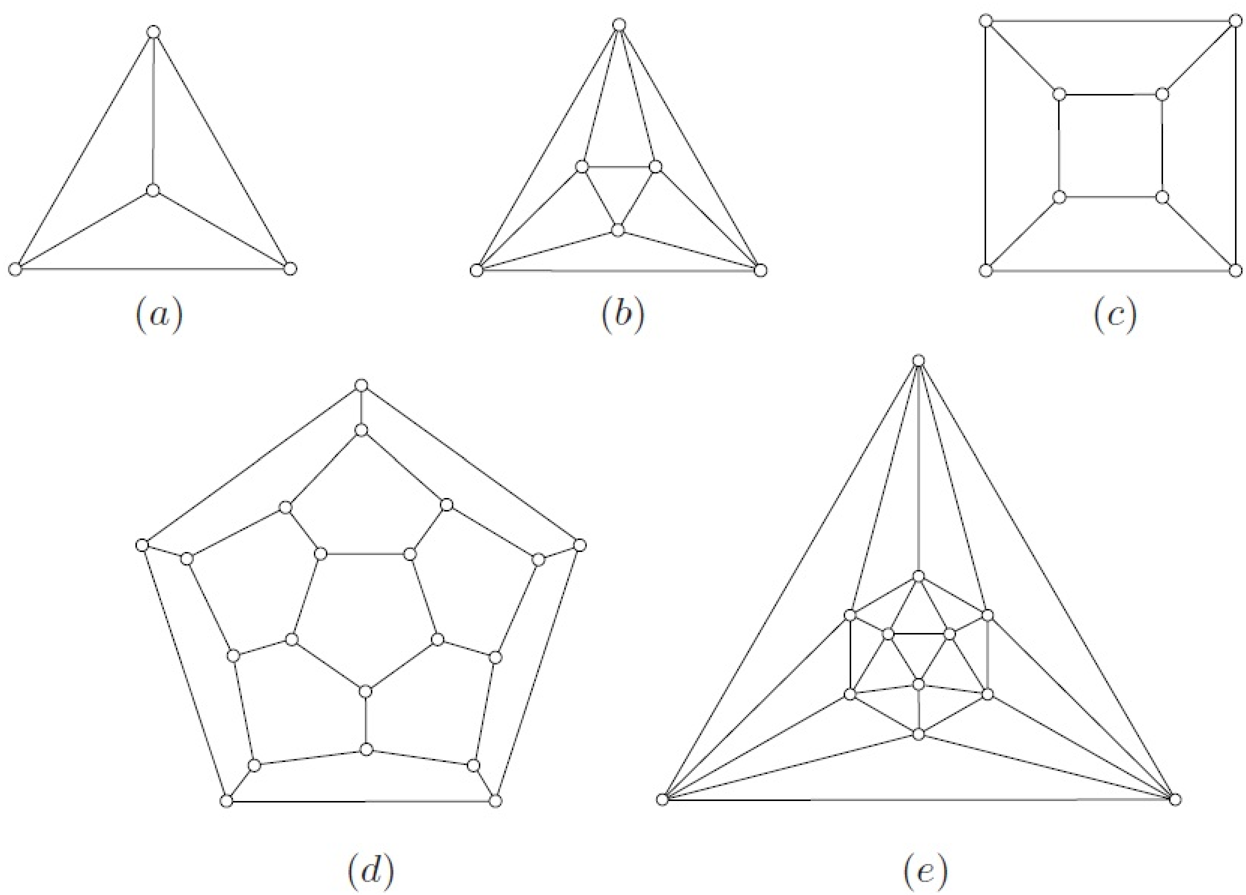


Figure 1.14. The five platonic graphs: (a) the tetrahedron, (b) the octahedron, (c) the cube, (d) the dodecahedron, and (e) the icosahedron.

Definition. A *set system* is an ordered pair $(V, \mathcal{F})$ where V is a set of elements and $\mathcal{F}$ is a family of subsets of V. A set system is also called a *hypergraph* where the elements of V are called *vertices* of the hypergraph and the elements of $\mathcal{F}$ are called *edges* or *hyperedges*. A hypergraph is *k-uniform* if each edge is a set of cardinality k.

Note. In set system $(V, \mathcal{F})$ where $\mathcal{F}$ consists of subsets of V each of cardinality two, we can interpret V as the vertex set of a loopless graph and $\mathcal{F}$ as the edge set (or multiset) of the graph. In this sense, graphs are special cases of hypergraphs.

Definition. A *geometric configuration* $(P, \mathcal{L})$ consists of a finite set P of elements called *points* and a finite family $\mathcal{L}$ of subsets of P, called *lines*, with the property that at most one line contains any given pair of points.

Note. Figure 1.15 gives drawings of the *Fano plane* and the *Desargue configuration*. In both, a line consists of 3 points so each represents a 3-uniform hypergraph. The Fano plane is the simplest example of a *projective plane* (see Exercise 1.3.13 and Chapter 7 of my notes for Design Theory).

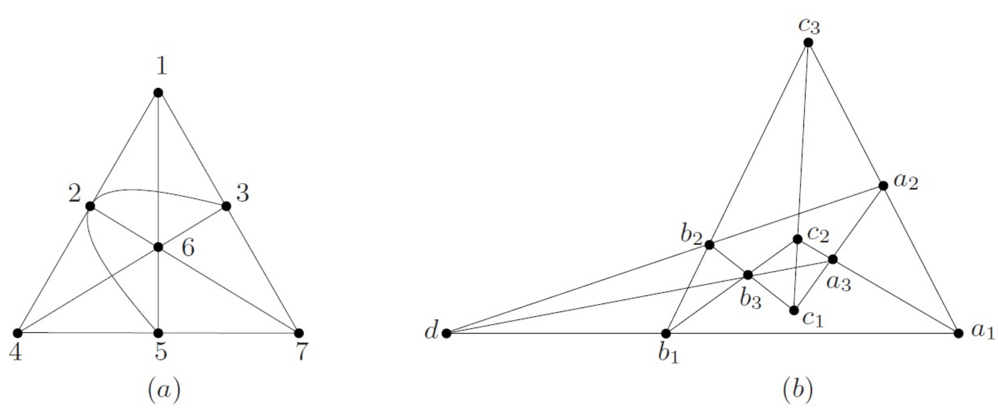


Figure 1.15. (a) The Fano plane, and (b) the Desargue configuration.

Definition. Let $H = (V, \mathcal{F})$ be a set system and define the bipartite graph $G[V, \mathcal{F}]$ where $v \in V$ and $F \in \mathcal{F}$ are adjacent if $v \in F$. $G[V, \mathcal{F}]$ is called the *incidence graph* of set system H. The bipartite adjacency matrix of $G[V, \mathcal{F}]$ is the *incidence matrix* of H.

Note. The incidence graph of the Fano plane is given in Figure 1.16; this graph is called the Heawood graph.

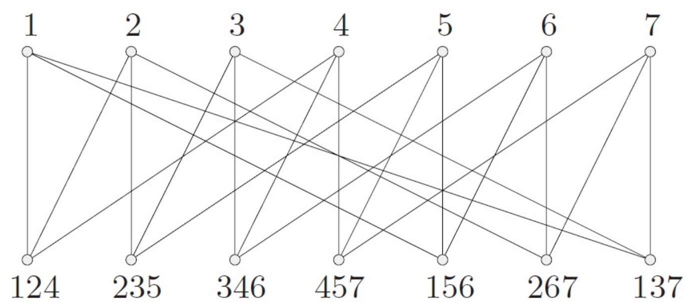


Figure 1.16

Definition. Let $(V, \mathcal{F})$ be a set system and define a graph with vertex set $\mathcal{F}$ and with two sets in $\mathcal{F}$ being adjacent if their intersection is nonempty. This is the *intersection graph* of the set system. If set system $(V, \mathcal{F})$ represents a loopless graph G (so that $\mathcal{F}$ is the edge set of the graph) then the corresponding intersection graph is called the *line graph* of G, denoted $L(G)$.

Note. Figure 1.17 gives a graph G and its line graph $L(G)$. Line graphs are further explained in Exercises 1.3.2, 1.3.3, and 1.3.4. Line graphs are further explored in my online notes for Algebraic Graph Theory on Section 1.7. Line Graphs.

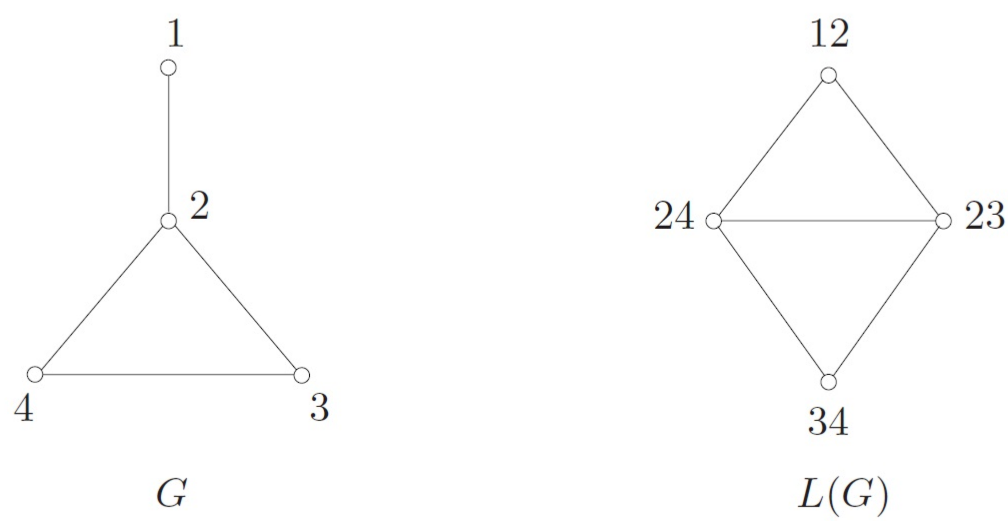


Figure 1.17

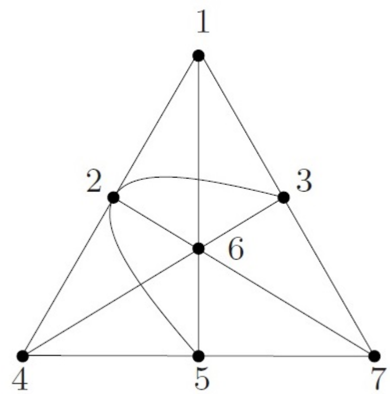
Definition. Let $(V, \mathcal{F})$ be a set system where $V = \mathbb{R}$ and $\mathcal{F}$ is a set of closed intervals in $\mathbb{R}$. The intersection graph of $(V, \mathcal{F})$ is an *interval graph*.

Note. Consider the set system $(\mathbb{R}, \mathcal{F})$, where $\mathcal{F} = \{[i, i+1] \mid i \in \mathbb{Z}\}$. For $i \neq j$, we have $[i, i+1] \cap [j, j+1] \neq \varnothing$ only when $i = j+1$ or $j = i+1$. So the interval graph based on the set sytem is an infinite path.

Note. The following definition is given in Exercise 1.3.6.

Definition. Let $H = (V, \mathcal{F})$ be a hypergraph. For $v \in V$, let $\mathcal{F}_v$ denote the set of edges of H incident to v. The *dual* of H is the hypergraph H^* whose vertex set is $\mathcal{F}$ and whose edges are the sets $\mathcal{F}_v$ where $v \in V$ (so edges $\mathcal{F}_u$ and $\mathcal{F}_v$ in H^* are adjacent if and only if vertices u and v are adjacent in H). A hypergraph is *self-dual* if it is isomorphic to its dual.

Note. Consider the hypergraph H of the Fano plane, as given in Figure 1.15(a).



The hyperedges of H are:

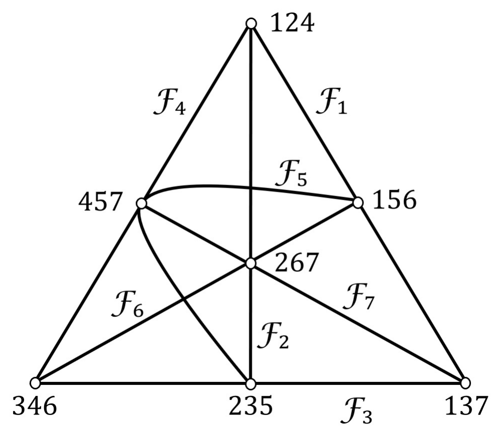
$$\mathcal{F}_1 = \{\{1,2,4\},\{1,3,7\},\{1,5,6\}\}, \mathcal{F}_2 = \{\{1,2,4\},\{2,3,5\},\{2,6,7\}\},$$

$$\mathcal{F}_3 = \{\{1,3,7\},\{2,3,5\},\{3,4,6\}\}, \mathcal{F}_4 = \{\{1,2,4\},\{3,4,6\},\{4,5,7\}\},$$

$$\mathcal{F}_5 = \{\{1,5,6\},\{2,3,5\},\{4,5,7\}\}, \mathcal{F}_6 = \{\{1,5,6\},\{2,6,7\},\{3,4,6\}\},$$

$$\mathcal{F}_7 = \{\{1,3,7\},\{2,6,7\},\{4,5,7\}\}.$$

The dual graph, H^*, is then the following:



We see from this that the Fano plane is a self dual hypergraph. The incidence matrix of the Fano plane, $\mathbf{M}_H$, is:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c} \begin{array}{ccccccc} 124 & 137 & 156 & 235 & 267 & 346 & 457 \end{array} \\
\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} = \mathbf{M}_H$$

The incidence matrix of the dual graph H^*, $\mathbf{M}_{H^*}$, is:

$$\begin{array}{c} \\ 124 \\ 137 \\ 156 \\ 235 \\ 267 \\ 346 \\ 457 \end{array}
\begin{array}{c} \begin{array}{ccccccc} \mathcal{F}_1 & \mathcal{F}_2 & \mathcal{F}_3 & \mathcal{F}_4 & \mathcal{F}_5 & \mathcal{F}_6 & \mathcal{F}_7 \end{array} \\
\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array} = \mathbf{M}_{H^*}$$

Notice that $\mathbf{M}_H \neq \mathbf{M}_{H^*}$ (in fact, it is somewhat of a coincidence that they are the same size), but that these matrices are transposes of each other, $\mathbf{M}_H = \mathbf{M}_{H^*}^t$. This is not a coincidence, and it is to be shown that this equality holds for any hypergraph H in Exercise 1.3.6(a).

Note. The following definition appears in Exercise 1.3.18.

Definition. Let Γ be a group, and let S be a set of elements of Γ not including the identity element. Suppose, furthermore, that the inverse of every element of S also belongs to S. The *Cayley graph* of Γ with respect to S is the graph $CG(\Gamma, S)$ with vertex set Γ in which two vertices x and y are adjacent if and only if $xy^{-1} \in S$.

Note. In Exercise 1.3.18(b) it is to be shown that every Cayley graph is vertex transitive, and in Exercise 1.3.18(c) it is to be shown (using the Petersen graph) that not every vertex transitive graph is a Cayley graph.

Note. As an example of a Cayley graph, consider the additive group $\Gamma = \mathbb{Z}$ and let $S = \{-1, 1\}$. Notice that for $x, y \in \mathbb{Z}$, we have $xy^{-1} = x + (-y) \in \{-1, 1\}$ implies that either $x = y - 1$ or $x = y + 1$. Hence, for every $n \in \mathbb{Z}$ we have that n is adjacent to both $n - 1$ and $n + 1$ and to no other vertices. Therefore, the Cayley graph of $\Gamma = \mathbb{Z}$ with respect to $S = \{-1, 1\}$ is an infinite path.

Note. The following definition appears in Exercise 1.3.19. It gives a family of Cayley graphs with respect to certain sets.

Definition. A *circulant* is a Cayley graph $\mathrm{CG}(\mathbb{Z}_n, S)$, where $\mathbb{Z}_n$ is the additive group of integers modulo n.

Note. Circulant graphs are explored in my online notes for Algebraic Graph Theory on Section 1.5. Circulant Graphs. In particular, the following example is given those notes.

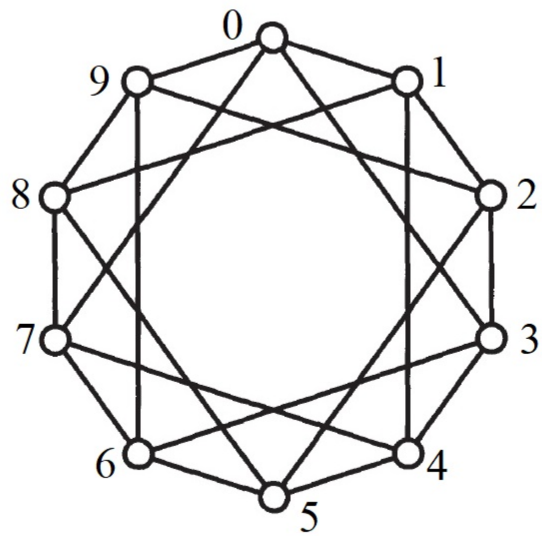


Figure 1.7. The circulant $X(\mathbb{Z}_{10}, \{-1, 1, -3, 3\})$

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