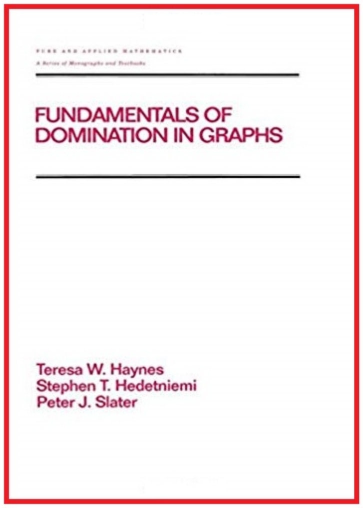
Section 1.5. Directed Graphs

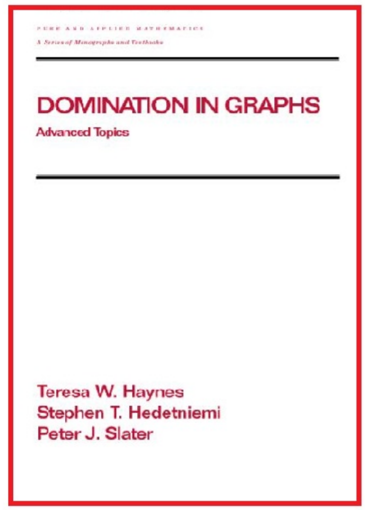
Note. In this section we define directed graphs and extend several concepts from graphs to directed graphs. These notes also include the definition of mixed graphs (not included in Bondy and Murty's text).

Definition. A *directed graph* D is an ordered pair $(V(D), A(D))$ consisting of a set $V = V(D)$ of *vertices*, set $A = A(D)$ (which is disjoint from $V(D)$) of *arcs*, and an *incidence function* $\psi_D : A(D) \to V(D) \times V(D)$ (so ψ_D associates arcs with ordered pairs of vertices). If a is an arc and $\psi_D(a) = (u, v)$, then a is said to *join* u *to* v, vertex u is the *tail* of a, vertex v is the *head* of a, and u and v are the *ends* of a. The number of arcs in D is denoted $a(D)$. The vertices which are joined to vertex v are is *in-neighbors* and the vertices to which vertex v is joined are its *out-neighbors.* The set of in-neighbors of vertex v in directed graph D is the *in-neighborhood* of v, denoted $N_D^-(v)$, and the set of out-neighbors of vertex v in D is the *out-neighborhood* of v, denoted $N_D^+(v)$. We often use the term *digraph* instead of "directed graph." A *strict digraph* is one with no loops (here, a loop is an arc with the same ends) nor parallel arcs (i.e., arcs with the same head and the same tail).

Note. In Bondy and Murty's text, if there is an arc in a digraph which joins vertex u to vertex v then they say that "u dominates v." We avoid the use of this term because of the similarity to the term "dominating set" and "domination," which are popular contemporary areas of research. See, for example, Exercise 13.2.12 and

Fundamentals of Domination in Graphs and *Domination in Graphs: Advanced Topics* by Teresa Haynes, Stephen Hedetniemi, and Pete Slater (Chapman & Hall/CRC Pure and Applied Mathematics, 1998).





Definition. For digraph $D = (V(D), A(D))$, define graph $G = (V(G), E(G))$ where $V(G) = V(D)$ and for every arc $a \in A(D)$ with $\psi_D(a) = (u, v)$ we have edge $e \in E(G)$ with $\psi_G(e) = uv$. Graph G is the *underlying graph* of D, denoted $G(D)$. Similarly (and a bit informally), we can create a digraph from a graph G by replacing each edge $\{u, v\}$ of G with the two arcs (u, v) and (v, u). The resulting digraph is the *associated digraph* of G, denoted $D(G)$. If a digraph is created from graph G by replacing each edge $\{u, v\}$ of G with one of the arcs (u, v) or (v, u), the resulting digraph is called an *orientation* of G, sometimes denoted $\vec{G}$. An orientation of a simple graph is an *oriented graph*. An orientation of a complete graph is a *tournament*.

Note. The idea behind a tournament, is that if n teams play each other exactly once in a "round-robin" tournament, then with the teams represented as vertices, outcomes of each match can be represented by an arc from the winner to the looser. We can make drawings of digraphs by representing arcs with arrows. Figure 1.25 gives such drawings for four unlabeled tournaments on four vertices.

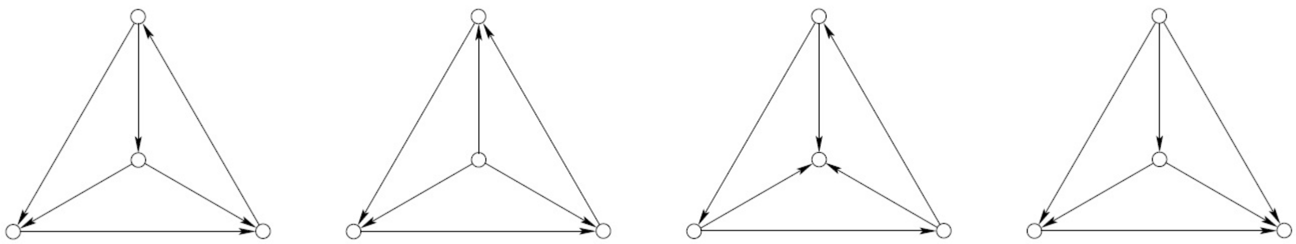


Figure 1.25

A common convention when drawing a digraph is to replace two oppositely oriented arcs between vertices u and v with a single edge between u and v (see page 33). This is done in Figure 1.26(b). We avoid this convention in these notes because we also want to address "mixed graphs" which have both edges and arcs.

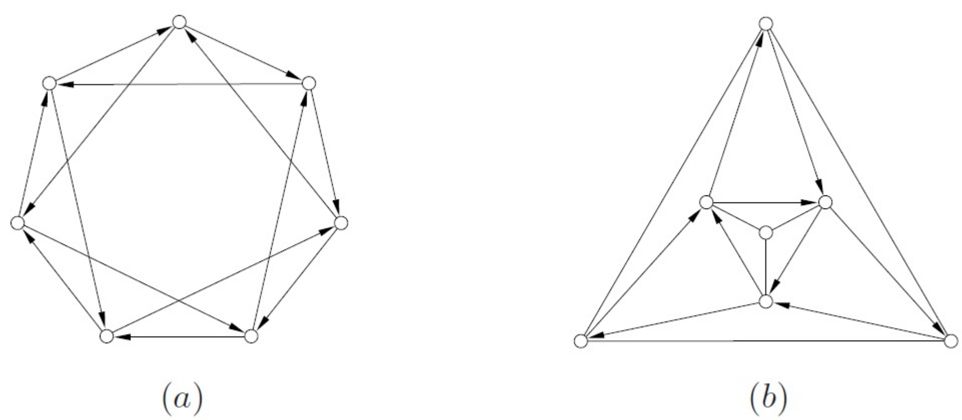


Figure 1.26. (a) The Koh-Tindell digraph, and (b) a directed analogue of the Petersen graph.

Note. Many of the concepts we have thus far defined for graphs carry over to digraphs, as follows.

Definition. The *degree* of a vertex in digraph D is the degree of v in the underlying graph $G(D)$. A digraph is *connected* if $G(D)$ is connected. The *indegree* of a vertex v in digraph D is the number of arcs with head v, denoted $d_D^-(v)$. The *outdegree* of a vertex v in digraph D is the number of arcs with tail v, denoted $d_D^+(v)$. The minimum indegree and outdegree of digraph D are denoted $\delta^-(D)$ and $\delta^+(D)$, respectively. The maximum indegree and outdegree of a digraph D are denoted $\Delta^-(D)$ and $\Delta^+(D)$, respectively. A digraph is *k-diregular* if each indegree and each outdegree is equal to k. A vertex of indegree 0 is a *source*, and a vertex of outdegree 0 is a *sink*. A *directed path* or *directed cycle* is an orientation of a path or cycle in which each vertex is joined to its successor in the sequence.

Note. The Koh-Tindell digraph of Figure 1.26(a) is 2-diregular and the "directed analogue of the Petersen graph" of Figure 1.26(b) is 3-diregular.

Definition. For digraph D, the *converse* is the digraph obtained from D by reversing the direction of each arc, and is denoted $\overleftarrow{D}$.

Note. The converse of the converse of a digraph is the original digraph itself. This observation (and the definition of converse) inspires Bondy and Murty to state (see page 33):

The Principle of Directional Duality.

Any statement about a digraph has an accompanying 'dual' statement, obtained by applying the statement to the converse of the digraph and reinterpreting it in terms of the original digraph.

For example, informally, whatever role indegree plays in digraph D, outdegree plays in the converse $\overleftarrow{D}$.

Note. The notes for the remainder of this section are not based on Bondy and Murty.

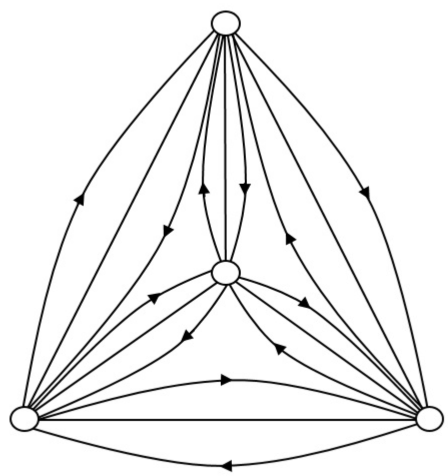
Definition. The *complete digraph* on n vertices, denoted D_n, is the digraph where $|V(D_n)| = n$ and for each $A(D_n) = \{(u, v) \mid u, v \in V(D_n), u \neq v\}$ (here we don't distinguish between an arc and its image under the incidence incidence function ψ). The *complete bipartite digraph*, denoted $D_{m,n}$, is the digraph where $V(D_{m,n}) = X \cup Y$ where $|X| = m$, $|Y| = n$, $X \cap Y = \varnothing$, and $A(D_{m,n}) = \{(x, y), (y, x) \mid x \in X, y \in Y\}$.

Definition. A *mixed graph* M is an ordered triple $(V(M), E(M), A(M))$ where $V(M)$ is a set whose elements are called vertices, $E(M)$ is a set disjoint from $V(M)$ whose elements are called *edges*, $A(M)$ is a set disjoint from both $V(M)$ and $E(M)$, together with *edge incidence function* ψ_M^E that associates with each edge of M an unordered pair of (not necessarily different) vertices u and v, and an *arc incidence function* ψ_M^A that associated with each arc of M an ordered pair of (not necessarily different) vertices u and v.

Note. We can denote an edge e of a mixed graph as $\{u, v\}$ where $\psi_M^E(e) = \{u, v\}$ and we denote the arc a of a mixed graph as (u, v) where $\psi_M^A(a) = (u, v)$. We use this notation in the following definition.

Definition. The *complete mixed graph* on n vertices, denoted M_n, is the digraph where $|V(M_n)| = n$, $E(M_n) = \{\{u, v\} \mid u, v \in V(M_n), u \neq v\}$, and $A(M_n) = \{(u, v) \mid u, v \in V(M_n), u \neq v\}$.

Note. The complete mixed graph M_4 can be drawn as:



Revised: 9/17/2022