Section 1.6. Infinite Graphs

Note. The primary topic of this course is finite graphs (and digraphs). We briefly describe a few infinite graphs in this section.

Definition. A graph $G = (V(G), E(G))$ is *infinite* if either $V(G)$ or $E(G)$ is infinite. An infinite graph is *countable* if both its vertex set and edge set are countable.

Note. An example of an uncountable graph is the simple graph $G_1 = (V(G_1), E(G_1))$ where $V(G_1) = \mathbb{R}$ and $\{x, y\} \in E(G_1)$ if and only if $|x - y| = 1$. Notice that this is a 2-regular graph. A related example is $G_q = (V(G_q), E(G_q))$ where $V(G_q) = \mathbb{R}$ and $\{x, y\} \in E(G_q)$ if and only if $x - y \in \mathbb{Q} \setminus \{0\}$. In this case, each vertex is (countably) infinite in degree.

Note. We briefly explore the graph G_q in more detail and relate it to a topic covered in Real Analysis 1 (MATH 5210). In G_q, xy is an edge if and only if $x - y \in \mathbb{Q} \setminus \{0\}$. Let $X = \mathbb{Q}$ and $Y = \mathbb{R} \setminus \mathbb{Q}$. We see that the every two vertices in X are adjacent (and so it forms a complete "subgraph" of G_q) and no vertex of X is adjacent to any vertex of Y. Hence, X and Y form a "separation" of G_q and, by definition, G_q is not connected. In Exercise 1.4, it is to be shown that any graph can be written as a disjoint union of connected graphs (called the *connected components* of the graph). This exercise is proved by introducing an equivalence relation for which the connected components are the equivalence classes. The complete graph on $\mathbb{Q}$ is one of these components of G_q. Since the vertex set is $\mathbb{R}$, then every real number appears in one of the components of G_q. One can show that each component of G_q is countable (using the fact that the rational numbers are countable) and that there are an uncountable number of components. These components are related to the construction of a non-Lebesgue-measurable set in Real Analysis 1. See my online notes on Section 2.6. Nonmeasurable Sets (Roydens 3rd Edition) for details. In these notes, an equivalence relation is introduced that yields equivalence classes that are the same as the connected components of graph G_q! One can show that there are infinitely many vertices of each component of G_q in the interval $[0, 1)$. A nonmeasurable set P is constructed by using the Axiom of Choice to create a set consisting of a single representative in $[0, 1)$ chosen from each component of G_q.

Note. Figure 1.27 gives drawings (well, partial drawings!) of the countable regular infinite graphs.

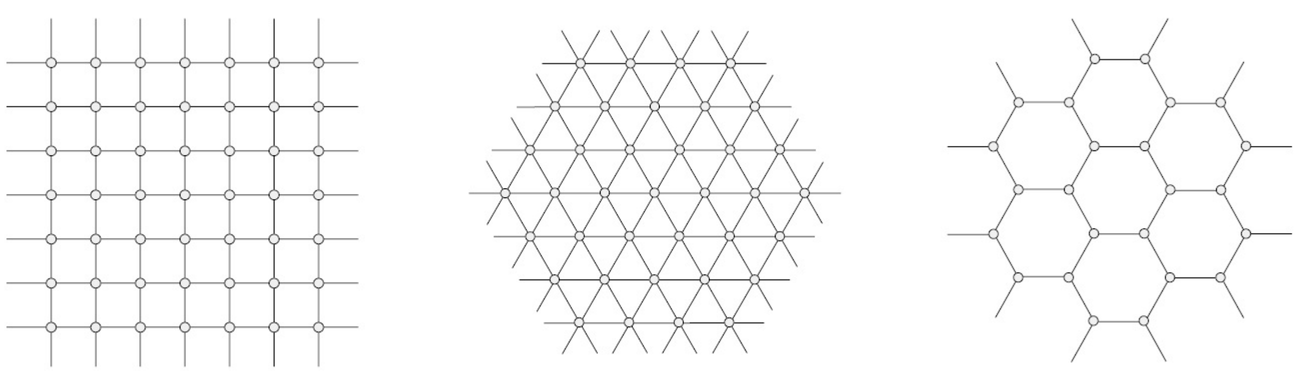

Figure 1.27. The square, triangular, and hexagonal lattices.

Definition. The *degree* of a vertex v in an infinite graph G is the cardinality of $N_G(v)$, denoted $d_G(v) = |N_G(v)|$. A *one-way infinite path* is an infinite countable simple graph whose vertices can be arranged in an infinite linear sequence, say $v_1, v_2, \ldots$, where v_i and v_j are adjacent if and only if either $i = j + 1$ or $j = i + 1$. A *two-way infinite path* is a countable simple graph whose vertices can be indexed by the integers $\mathbb{Z}$, say $\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots$, where v_i and v_j are adjacent if and only if either $i = j + 1$ or $j = i + 1$.

Note. The square lattice of Figure 1.27 is the Cartesian product of two-way infinite paths.

Note. Reinhard Diestel's *Graph Theory* (Graduate Texts in Mathematics 173), 5th edition (Springer, 2016) includes a chapter (Chapter 8) on infinite graphs. An online version can be previewed at Diestel's *Graph Theory* website, which includes Chapter 8. You can get online access to the 2nd edition of this through the ETSU library. An older book by the same author is *Graph Decompositions: A Study in Infinite Graph Theory* (Oxford Science Publications, 1990).

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