# Chapter 10. Planar Graphs

**Note.** Some of the material in this chapter requires results from other areas of math, namely topology. We refer to the Jordan Curve Theorem in Section 10.1 (a result concerning simple closed curves in the plane) and to the Classification Theorem for Surfaces in Section 10.6 (when we consider embedding graphs in surfaces).

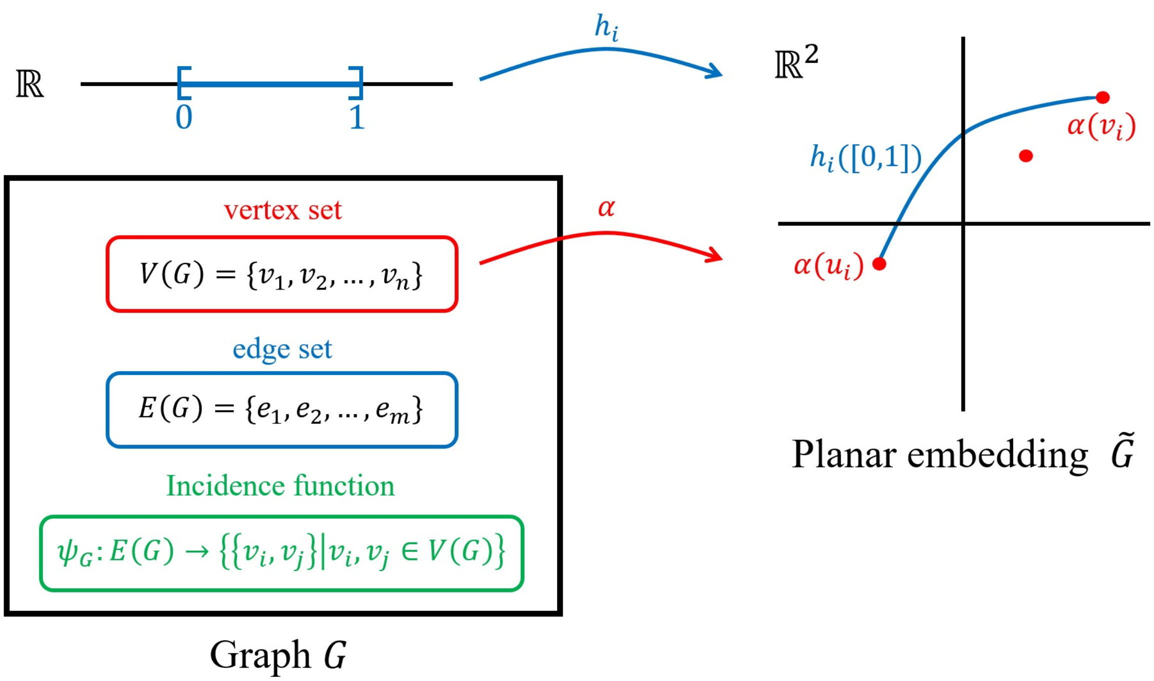
## Section 10.1. Plane and Planar Graphs

**Note.** We borrow some results from topology in this section and somewhat reduce our level of rigor. In an attempt to maximize what rigor we *do* have, we state a formal definition of a planar embedding in terms of homeomorphisms (in Note 10.1.A; this material is not in the text book).

**Definition.** A graph is *embeddable in the plane*, or *planar*, if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing is a *planar embedding* of the graph and is itself called a *plane graph*.

**Note.** A planar graph is simply a graph which *can* be embedded in the plane with no edge crossings. A "plane graph" has more structure than a graph since it reflects some specific embedding in the plane; a planar graph can be associated with several different plane graphs (the plane graphs will be isomorphic as graphs, but may have different properties as embeddings in the plane). The topic of crossing number is covered in Introduction to Graph Theory (MATH 4347/5347); see my online notes for this class on Section 9.1. Crossing Number.

**Note 10.1.A.** More formally, a *planar embedding* of a (finite) planar graph $G$ is a one-to-one (injective) mapping $\alpha$ of $V(G)$ into $\mathbb{R}^2$ and a collection of $m$ homeomorphisms (defined below in Note 10.1.C), $h_1, h_2, \ldots, h_m$, mapping $[0,1]$ into $\mathbb{R}^2$ such that for edge $e_i$ of $G$ with $\psi_G(e_i) = u_i v_i$, where $\psi_G$ is the incidence function of $G$, we have homeomorphism $h_i$ with $h_i(0) = \alpha(u_i)$, $h_i(1) = \alpha(v_i)$, $h_i(x) \notin V(G)$ for $x \in (0,1)$, and for any $i \neq j$ we have that $\{h_i(x) \mid x \in (0,1)\}$ and $\{h_j(x) \mid x \in (0,1)\}$ are disjoint. In this way, each vertex of $G$ is represented by a point in $\mathbb{R}^2$, each edge is represented by an "arc" in $\mathbb{R}^2$ which is a homeomorphic image of $[0,1]$ that joins the ends of the edge, an arc representing an edge only contains images of vertices which are ends of the edge, and any pair of arcs intersect only at their ends (if at all).




**Note.** We denote a planar embedding of a planar graph $G$ as $\tilde{G}$. We call the images of the vertices under $\alpha$ the *points* of $\tilde{G}$ and the sets $\{h_i(x) \mid x \in [0,1]\}$ the *lines* of $\tilde{G}$.

**Note 10.1.B.** The term "line" above is appropriate since every simple planar graph can be embedded in the plane where each "line" is actually a line segment. This was shown by K. Wagner in "Bemerkungen zum Vierfarbenproblem [Comments on the Four-Color Problem]," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, **46**, 26–32 (1936). A copy is available online (in German, of course). See also Exercise 10.1.6.

**Note.** We now give some topological definitions and one theorem concerning the plane $\mathbb{R}^2$ (treated as a topological space under the "usual" topology). More precise definitions can be found in James R. Munkres' *Topology*, 2nd edition, Prentice Hall (2000) (in particular, see "Chapter 10. Separation Theorems in the Plane"). See also my online notes Algebraic Topology for some relevant notes based on Munkres' book. Here, we largely follow Bondy and Murty but make a few small changes motivated by Munkres.

**Note 10.1.C.** A *homeomorphism* between topological spaces is a continuous mapping between the spaces which has a continuous inverse (notice that since a homeomorphism is invertible by definition, then it is necessarily one-to-one). A *simple curve* in $\mathbb{R}^2$ is a homeomorphic image of $[0, 1]$ in $\mathbb{R}^2$. A *simple closed curve* in $\mathbb{R}^2$ is a homeomorphic image of the circle $S^2 = \{x^2 + y^2 = 1 \mid (x, y) \in \mathbb{R}^2\}$ in $\mathbb{R}^2$. (Bondy and Murty require only continuity and not a homeomorphism in defining a curve, so they can streamline the definition of a simple closed curve. But the proof of the main theorem we need as background is dependent on the use of homeomorphisms.) Given two points $x$ and $y$ in a topological space $X$, a *path* from $x$ to $y$ is a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. A space $X$

is *path connected* if every pair of points of $X$ can be joined by a path in $X$. See my online notes on for Topology [MATH 4357/5347] on 3.24. Connected Subspaces of the Real Line. Bondy and Murty use the term "arcwise-connected" for path connected; of course Bondy and Murty have a different use for the term "path."

**Theorem 10.1.** The Jordan Curve Theorem (Bondy and Murty's version)
Any simple closed curve $C$ in the plane $\mathbb{R}^2$ partitions the rest of the plane $\mathbb{R}^2 \setminus C$ into two disjoint arcwise-connected open sets.

**Note.** Munkres states a related result:

> **Theorem 61.3.** The Jordan Separation Theorem.
>
> Let $C$ be a simple closed curve in $S^2$ (the surface of a locally 2-dimensional sphere). Then $C$ separates $S^2$ (in the sense of a topological separation mentioned in our Section 1.1 in Note 1.1.B).

The reason that Munkres can state the theorem in terms of $S^2$ instead of $\mathbb{R}^2$ will become clear when we look at stereographic projection later in this section.

**Definition.** The two open sets into which a simple closed curve $C$ partitions the plane are called the *interior* and the *exterior* of $C$, denoted $\text{int}(C)$ and $\text{ext}(C)$, respectively. The topological closure of $\text{int}(C)$ is denoted $\text{Int}(C)$ and the topological closure of $\text{ext}(C)$ is denoted $\text{Ext}(C)$.

**Note.** Recall that a continuous image of a compact set is compact, so a simple closed curve in $\mathbb{R}^2$ is a compact set in $\mathbb{R}^2$. By the Heine-Borel Theorem (see Theorem 3-10/3-11 in my online Analysis 1 [MATH 4217/5217] notes on 3-1. Topology of the Real Numbers) this means that a simple closed curve is a topologically closed and bounded set in $\mathbb{R}^2$. So one of the two open sets which partition $\mathbb{R}^2 \setminus C$ must be bounded and the other is unbounded. This is how we distinguish between $\text{int}(C)$ and $\text{ext}(C)$. Munkres proves this in Theorem 63.4, "The Jordan Curve Theorem."

**Note.** The Jordan Curve Theorem implies that an arc (or, in the topological sense, "path") joining a point of $\text{int}(C)$ to a point of $\text{ext}(C)$ must meet $C$ in at least one point:

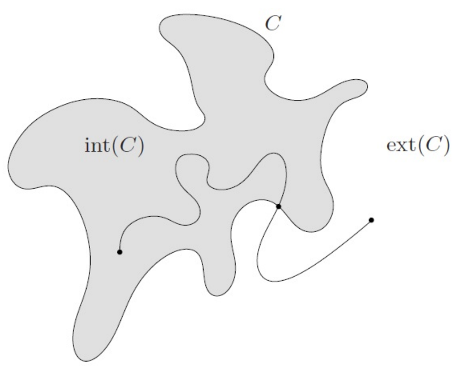



**Figure 10.2**

This property is how we will use the Jordan Curve Theorem to show that certain graphs are NOT planar.

**Theorem 10.2.** $K_5$ is nonplanar.

**Note.** The Jordan Curve Theorem can also be used to prove that $K_{3,3}$ is nonplanar, as is to be done in Exercise 10.1.1(b). In Theorem 10.30, "Kuratowski's Theorem," we'll see that $K_5$ and $K_{3,3}$ are fundamental nonplanar graphs.

**Definition.** A graph derived from graph $G$ by a finite sequence of edge subdivisions is a *subdivision* of $G$.

**Note.** Subdivisions of $K_5$ and $K_{3,3}$ are given in Figure 10.4:

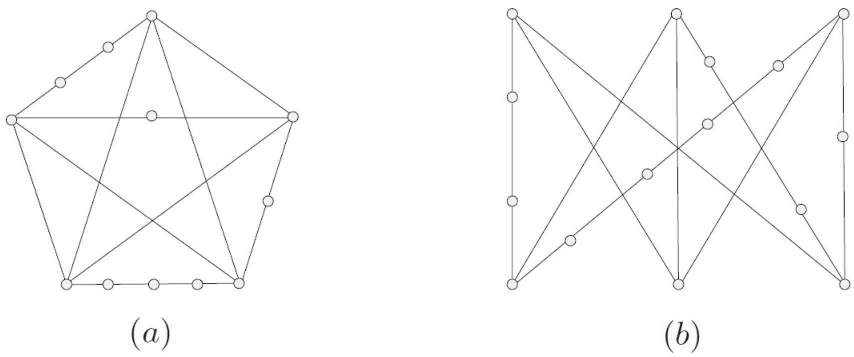


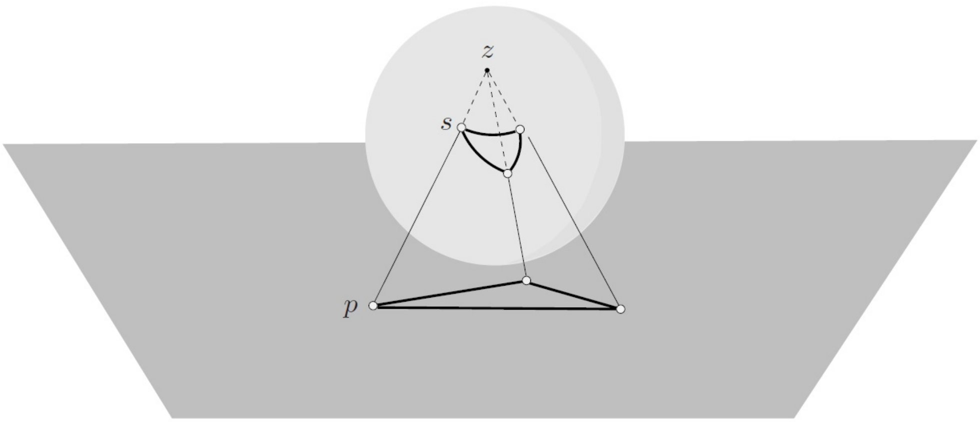

**Figure 10.4**

Notice that these two subdivided graphs are not planar.

**Proposition 10.3.** A graph $G$ is planar if and only if every subdivision of $G$ is planar.

**Note.** The proof of Proposition 10.3 is to be given in Exercise 10.1.2. Since $K_5$ and $K_{3,3}$ are nonplanar, this result implies that no planar graph can contain a subdivision of either $K_5$ and $K_{3,3}$. Kuratowski's Theorem (Theorem 10.30, proved in 1930) is the converse of this: any nonplanar graph contains a subdivision of either $K_5$ or $K_{3,3}$.

**Note 10.1.D.** Consider a sphere $S$ resting on a plane $P$ (i.e., $P$ is tangent to $S$), and denote by $z$ the point on $S$ diametrically opposite the point of tangency. Define a map $\pi : S \setminus \{z\} \to P$ defined by $\pi(s) = p$ if and only if the points $z$, $s$, and $p$ are collinear. This mapping is called a *stereographic projection* from $z$:




Stereographic projection also arises in the setting of complex analysis. It can be used to define the extended complex plane $\mathbb{C}_\infty$ and used to define a metric on $\mathbb{C}_\infty$; see my online notes for Complex Analysis 1 (MATH 5510) on I.6. The Extended Plane and Its Spherical Representation and Supplement. The Extended Complex Plane.

**Note.** Stereographic projection can be used to prove the following, which gives an equivalence of planar graphs with graphs embeddable on the sphere.

**Theorem 10.4.** A graph $G$ is embeddable in the plane if and only if it is embeddable in the sphere.

*Revised: 3/12/2023*