# **Section 10.2.** Duality

**Note.** We introduce the idea of a face of a planar graph and define the dual $G^*$ of plane graph $G$. Several properties of the dual graph are given.

**Definition.** By the Jordan Curve Theorem (Theorem 10.1), a plane embedding of a planar graph $G$ partitions the rest of the plane (that is, the complement of the points and lines of the embedding) into a finite number of arcwise-connected open sets called the *faces* of $G$. The unbounded face is called the *outer face*. The set of faces is denoted $F(G)$ and the number of faces is denoted $f(G)$.

**Note.** The idea of a face of a graph extends to the setting of embeddings on surfaces (though we may loose the idea of an outer face on finite surfaces). Figure 10.7 gives a plane embedding of a graph with five faces:

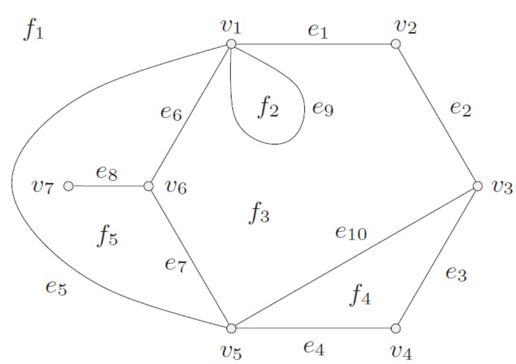



**Figure 10.7**

**Definition.** The *boundary of a face* $f$ is the boundary of the open set $f$ in the "usual topological sense," denoted $\partial(f)$. Face $f$ is said to be *incident* with the vertices and edges in $\partial(f)$ and two faces are said to be *adjacent* if their boundaries share an edge.

**Proposition 10.5.** Let $G$ be a planar graph and let $f$ be a face in some planar embedding of $G$. Then $G$ admits a planar embedding whose outer face has the same boundary as $f$.

**Note.** Bondy and Murty state on page 250: "In the ensuing discussion of plane graphs, we assume, without proof, a number of other intuitively obvious statements concerning their faces." Namely, we assume:

• A planar embedding of a tree has just one face.

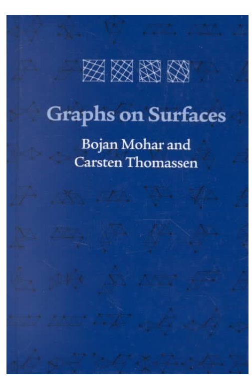
• Each face boundary in a connected plane graph is itself connected.

• Any point $p$ on a simple closed curve $C$ can be connected to any point not on $C$ be means of a simple curve which meets $C$ only at $p$.

We will use this third property below when giving some simplifying properties of dual edges (in Note 10.2.B). Bondy and Murty mention the following as useful in establishing some of these results:

> **Theorem 10.6.** THE JORDAN SCHÖNFLIESS THEOREM. Any homeomorphism of a simple closed curve in the plane onto another simple closed curve can be extended to a homeomorphism of the plane.

For a rigorous approach, Bondy and Murty recommend B. Mohar and C. Thomassen's *Graphs and Surfaces*, Baltimore: Johns Hopkins University Press (2001). The ETSU Sherrod Library has a copy of this book (QA.166.M64.2001):



I have online notes based on this source for Topological Graph Theory.

**Definition.** An edge is said to *separate the faces* $f$ incident with it. The *degree of a face* $f$ is the number of edges in its boundary $\partial(f)$, with cut edges counted twice, and is denoted $d(f)$.

**Note 10.2.A.** A cut edge in a plane graph has just one incident face (and conversely); it could be for example an edge of a path in a face or it could be a cut edge with the outer face on "both sides" of it. In Figure 10.7, edge $e_9$ separates the faces $f_2$ and $f_3$ and the edge $e_8$ separates the face $f_5$ from itself. Note the degree of face $f_5$ is 5 since edge $e_8$ counts twice:

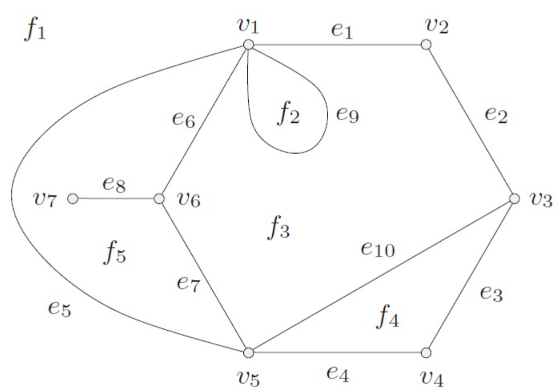



**Figure 10.7**

**Definition.** Let $G$ be a connected plane graph. To *subdivide a face* $f$ of $G$ is to add a new edge $e$ joining the two vertices on its boundary in such a way that $e$ lies entirely in the interior of $f$ except for the endpoints of $e$.

**Note.** Figure 10.8 shows that the subdivision of a face. Notice that the new graph $G + e$ has exactly one more face than $G$.

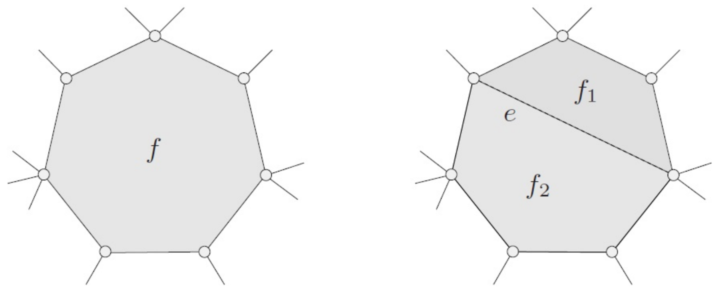



**Figure 10.8**

In Exercise 10.2.2, it is to be shown that: "The boundary of a face of a connected graph can be regarded as a closed walk in which each cut edge of the graph lying in the boundary is traversed twice." The following is due to H. Whitney (1932).

**Theorem 10.7.** In a nonseparable plane graph other than $K_1$ or $K_2$, each face is bounded by a cycle.

**Corollary 10.8.** In a loopless 3-connected plane graph, the neighbors of any vertex lie on a common cycle.

**Definition.** Let $G$ be a plane graph. For each face $f$ of $G$ define a vertex $f^*$ of graph $G^*$. Two vertices $f^*$ and $g^*$ are joined by the edge $e^*$ in $G^*$ if and only if their corresponding faces $f$ and $g$ are separated by edge $e$ in $G$. Graph $G^*$ is the *dual* of $G$.

**Note.** The definition of the dual $G^*$ of plane graph $G$ implies:

$$v(G^*) = f(G), \ \ e(G^*) = e(G), \ \text{and } d_{G^*}(f^*) = d_G(f) \text{ for all } f \in F(G). \tag{10.1}$$

**Note 10.2.B.** Informally, we place vertex $f^*$ in corresponding face $f$ of plane graph $G$ and then draw each edge $e^*$ in such a way that it crosses the corresponding edge $e$ of $G$ exactly once and intersects no other edges of $G$ (see Bondy and Murty page 252). If $e$ is a cut edge of $G$ then $e$ has the same face on both sides of $e$, and so $e^*$ in $G^*$ is a loop. Conversely, for each loop in $G^*$ there is a cut edge in $G$. The dual of the plane graph of Figure 10.7 is given in Figure 10.9:

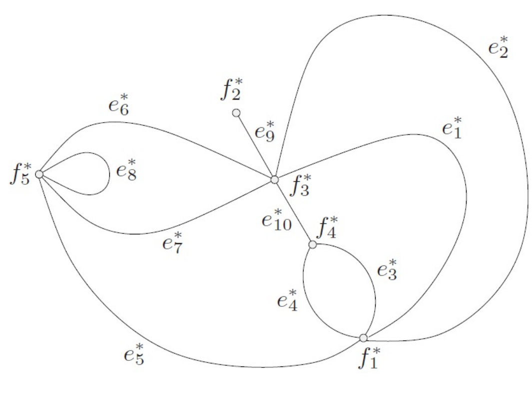



**Figure 10.9**

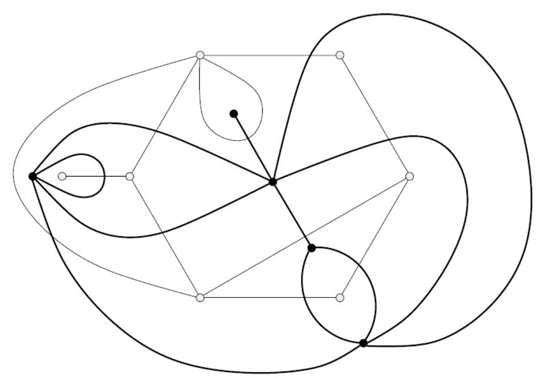


**Figure 10.10**

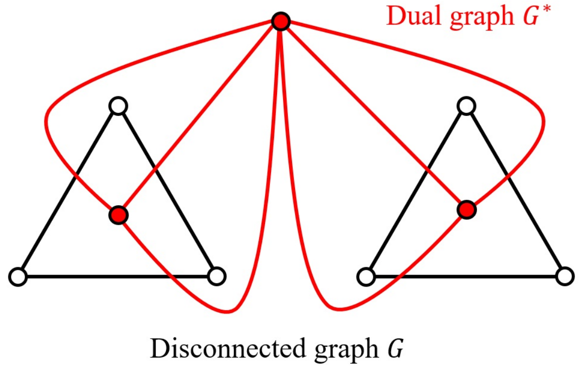
**Note.** Note 10.2.B helps us justify the following.

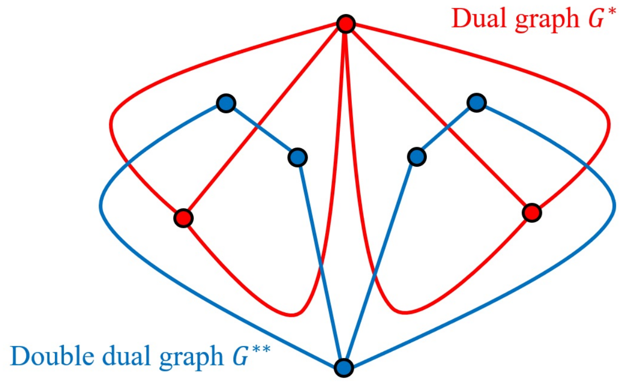
**Lemma 10.2.A.** The dual $G^*$ of a plane graph $G$ is itself a plane graph.

**Definition.** A drawing of $G^*$ for plane graph $G$ as described above is a *plane dual* of the plane graph $G$.

**Proposition 10.9.** A dual $G^*$ of a plane graph $G$ is connected.

**Note.** We can consider the dual of the dual of plane graph $G$ (the "double dual") of $G$, $G^{**}$. $G$ is connected if and only if $G^{**} \cong G$, as is to be shown in Exercise 10.2.4 (or Exercise 10.2.6, depending on the printing of the book). This is illustrated here for $G$ not connected:




Notice that $G$ is not connected, but that $G^*$ and $G^{**}$ are (as expected from Proposition 10.9). Notice that $G$, $G^*$, and $G^{**}$ each have six edges. $G$ has six vertices, but $G^{**}$ has only five vertices. In particular, $G \not\cong G^{**}$.

**Note.** A peculiar property of duals is that different planar embeddings of a given graph may yield different duals. In Figure 10.11, we see two different embeddings of the same planar graph (so for planar graph may have more than one planar embedding and more than one associated plane graph), and the duals of these two plane graphs are different. The plane graph on the left has two faces of degree 5 and the graph on the right has only one face of degree 5. As Bondy and Mutry state on page 253: "Thus the notion of a dual graph is meaningful only for plane graphs [i.e., where the embedding in the plane is given], and not for planar graphs [which *have* an embedding, but it is not explicitly given] in general." Bondy and Murty call the next result "a dual version of Theorem 1.1."

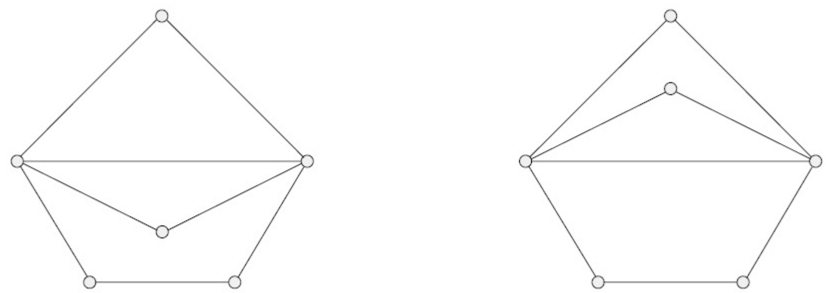


**Figure 10.11**

**Theorem 10.10.** If $G$ is a plane graph, then $\sum_{f \in F} d(f) = 2m$.

**Definition.** A simple connected graph in which all faces have degree three is a *plane triangulation*.

**Proposition 10.11.** A simple connected plane graph is a triangulation if and only if its dual is cubic.

**Note.** In Exercise 10.2.3, it is to be shown that every simple connected plane graph on $n \geq 3$ vertices is a spanning subgraph of a triangulation. We'll show in Section 10.3. Euler's Formula (see Corollary 10.21) that no simple spanning supergraph of a triangulation is planar. So triangulations are "maximal planar graphs" in terms of spanning subgraphs/supergraphs of planar graphs.

**Note 10.2.C.** If $G$ is a planar graph then for any edge $e$ of $G$ we have $G \setminus e$ is planar (just remove *line* $e$ from a plane embedding $\tilde{G}$ of $G$). In fact, if $G$ is a planar graph then the contraction of an edge yields a planar graph $G/e$, as is shown in Exercise 10.1.4(a). The next two results show relationships between edge contraction and edge deletion in plane graphs and their duals.

**Proposition 10.12.** Let $G$ be a connected plane graph and let $e$ be an edge of $G$ that is not a cut edge. Then $(G \setminus e)^* \cong G^*/e^*$.

**Proposition 10.13.** Let $G$ be a connected plane graph and let $e$ be a link (i.e., a nonloop) of $G$. Then $(G/e)^* \cong G^* \setminus e^*$.

**Note.** Bondy and Murty declare that the next result "turns out to be very useful."

**Theorem 10.14.** The dual of a nonseparable plane graph is nonseparable.

**Note.** We now define the dual of a digraph. In so doing, we need a well-defined way to orient the arcs of the dual.

**Definition.** Let $D$ be a plane digraph with underlying plane graph $G$. Then $G$ has a plane dual $G^*$. Let $a$ be an arc of $D$ that separates two faces of $G$. As $a$ is traversed from its tail to its head, one of these faces lies to the left of $a$, which we denote $l_a$, and one lies to the right of $a$, which we denote $r_a$ (if $a$ is a cut "edge" of $D$ then $l_a = r_a$). For each arc of $D$, orient the edge of $G^*$ that crosses this edge as an arc $a^*$ by designating the end of $a^*$ lying in $l_a$ as the tail of $a^*$ and the end lying in $r_a$ as its head. The resulting plane digraph $D^*$ is the *directed plane dual* of $D$.

**Note.** The ideas of "lies to the" left/right can be made more rigorous by giving specific homeomorphisms which map $[0,1]$ onto an arc of plane digraph $D$. For an example of this in the setting of the extended complex plane $\mathbb{C}_\infty$ (which is homeomorphic, under stereographic projection, to the 2-sphere) see my online notes for Complex Analysis 1 (MATH 5510) on III.3. Analytic Functions as Mapping, Möbius Transformations; in particular, see Definition III.3.20 and the definition that follows it. Figure 10.13 below shows the directed plane dual (in bold) of a plane digraph; notice the "left/right" induced directions on the dual digraph.

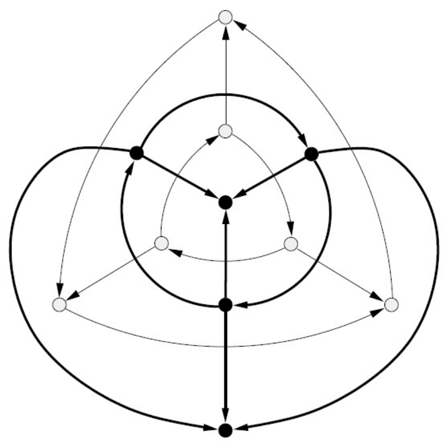


**Figure 10.13**

**Note.** In Section 2.6, "Even Subgraphs," we defined the *edge (vector) space* of graph $G$ over scalar field $GF(2) \cong \mathbb{Z}_2$ where the vectors are sets of edges of $G$, vector addition is defined in terms of symmetric differences, and scalar multiplication is defined as $0E = \varnothing$ and $1E = e$ for vector $E$. The *cycle (vector) space* of $G$ is the subspace of the edge space generated by cycles, and the bond space of $G$ is the subspace of the edge space generated by the bonds of $G$. We now consider how these spaces are related between $G$ and $G^*$.

**Proposition 10.15.** Let $G$ be a plane graph, $G^*$ a plane dual of $G$, $C$ a cycle of $G$, and $X^*$ the set of vertices of $G^*$ that lie in $\text{int}(C)$. Then $G^*[X^*]$ is connected.

**Note 10.2.D.** The proof of Proposition 10.15 is similar to the proof of Proposition 10.9 and is left as Exercise 10.2.A. The proof of Proposition 10.15 follows by deleting edges in $\text{Ext}(C)$, observing that this results in a plane graph with edge set $E(G[X])$ by Note 10.2.C, and applying Proposition 10.9 to this new plane graph. Notice that isolated vertices do not affect a plane graph.

**Note.** In the next result, we denote by $S^*$ the set $\{e^* \mid e \in S \subseteq E(G)\} \subseteq E(G^*)$ for graph $G$.

**Theorem 10.16.** Let $G$ be a connected plane graph, and let $G^*$ be a plane dual of $G$.

**(a)** If $C$ is a cycle of $G$, then $C^*$ is a bond of $G^*$.

**(b)** If $B$ is a bond of $G$, then $B^*$ is a cycle of $G^*$.

**Note.** The proof of part (b) is left as Exercise 10.2.9. The dual relationship between bonds and cycles given in Theorem 10.16 implies the following.

**Corollary 10.17.** For any plane graph $G$, the cycle space of $G$ is isomorphic to the bond space of $G^*$.

**Note.** Similar results hold for digraphs. Let $D$ be a plane digraph and $D^*$ its directed plane dual. Denote by $S^*$ the set $\{a^* \mid a \in S \subseteq A(D)\} \subseteq A(D^*)$. Part (b) of the following is the directed analogue of Corollary 10.17. The proof is to be given in Exercise 10.2.13.

**Theorem 10.18.** Let $D$ be a connected plane digraph and let $D^*$ be a plane directed dual of $D$.

**(a)** Let $C$ be a cycle of $D$, with a prescribed sense of traversal. Then $C^*$ is a bond $\partial(X^*)$ of $D^*$. Moreover the set of forward arcs of $C$ corresponds to the outcut $\partial^+(X^*)$ and the set of reverse arcs of $C$ to the incut $\partial^-(X^*)$.

**(b)** Let $B = \partial(X)$ be a bond of $D$. Then $B^*$ is a cycle of $D^*$. Moreover the outcut $\partial^+(X)$ corresponds to the set of forward arcs of $B^*$ and the incut $\partial^-(X)$ corresponds to the set of reverse arcs of $B^*$ (with respect to a certain sense of traversal of $B^*$).

*Revised: 3/12/2023*