Section 10.3. Euler's Formula

Note. You are probably familiar with Euler's formula which relates the numbers of vertices, edges, and faces of a connected plane graph (not to be confused with Euler's formula that relates complex exponential functions, sines, and cosines). It is useful in establishing when a graph is *not* planar and we use it in this section to show that neither K_5 nor $K_{3,3}$ are planar.

Note. Bondy and Murty state that Euler first established the formula for polyhedra graphs in 1752. Additional historical details can be found in Robin Wilson's *Four Colors Suffice: How the Map Problem Was Solved* (Princeton University Press, 2002); see page 47. Rene Descartes showed an interest in the total number of angles in all faces of a polyhedron with f faces and v solid angles, sometime around 1639. Leibniz copied this work in the 1670s but it was not known publicly until 1859, well after Euler's work. Euler gave a "proof by dissection" (as Wilson describes it) for polyhedra, which is just an inductive proof. Euler presented his proof to the Saint Petersburg Academy of Sciences on September 9, 1751 and published his two papers in the Academy's proceedings in 1752 (explaining the date given by Bondy and Murty) and in 1756. "Unfortunately for Euler, it is not at all obvious that his dissection procedure can always be carried out, so his proof must be regarded as deficient [Wilson, page 47]." Euler produced thousands of mathematical works (the creation of his collected works is an ongoing project today!) and he commonly took large intuitive leaps (such as "cancelling infinities") that would later be made rigorous. Adrien-Marie Legendre gave a correct proof of the formula in his 1794 *Eléments de geométrie*. So, as with many classical results, the story is more involved than the title of the result would suggest.

René Descartes (1596–1650)

Leonhard Euler (1707–1783)

Adrien-Marie Legendre (1752–1833)

These images are from the MacTutor History of Mathematics Archive. The caricature of Legendre is by J-L Boilly and is the only known portrait Legendre.

Theorem 10.19. Euler's Formula.
For a connected plane graph G, $v(G) - e(G) + f(G) = 2$.

Note. We have seen that a given planar graph can have plane embeddings which have different properties (see Figure 10.11 above). The next result gives one property shared by all planar embeddings of a given connected planar graph.

Corollary 10.20. All planar embeddings of a connected planar graph have the same number of faces.

Note. We will use the following to show K_5 and $K_{3,3}$ are nonplanar. We also use this this in our exploration of the Four Colour Theorem 11.4 (see Note 11.1.A).

Corollary 10.21. Let G be a simple planar graph on at least three vertices. Then $m \leq 3n - 6$. Furthermore, $m = 3n - 6$ if and only if every planar embedding of G is a triangulation.

Corollary 10.22. Every simple planar graph has a vertex of degree at most five.

Note. We now show that K_5 and $K_{3,3}$ are nonplanar.

Corollary 10.23. K_5 are nonplanar.

Corollary 10.24. $K_{3,3}$ is nonplanar.

Revised: 4/5/2020