Section 10.4. Bridges

Note. We introduce a structure associated with a connected graph and use it to show that a certain class of planar graphs have unique planar embeddings.

Definition. Let H be a proper subgraph of a connected graph G. The set of edges $E(G) \setminus E(H)$ can be partitioned into "classes" as follows:

1. For each component F of $G - V(H)$, there is a "class" consisting of the edges of F together with the edges linking F to H (these second type of edges are in $G[V(F), V(H)]$).

2. Each remaining edge e (that is, e had both ends in $V(H)$) defines a singleton "class" $\{e\}$.

For one of the classes K (so $K \subseteq E(G) \setminus E(H)$), the edge-induced subgraph of G induced by K, $G[K]$, is a *bridge* of H in G. For a bridge B of H, the elements of $V(B) \cap V(H)$ are its *vertices of attachment* of H; the remaining vertices of B are its *internal vertices*. A bridge is *trivial* if it has no internal vertices (and so is of the second class given above). A bridge with k vertices of attachment is s *k-bridge*. Two bridges with the same vertices of attachment are *equivalent bridges*.

Note. Bridges of H can intersect only in vertices of H. Any two vertices of a bridge of H are connected by a path in the bridge that is internally disjoint from H (trivially in the second class, and using two edges of $G[V(F), V(H)]$ and the fact that F is connected in the first class).

Note. In Figure 10.15, H is taken as a cycle and there are six bridges of H in G. Notice that we consider connected components of $G - V(H)$ and the trivial edges in $E(G) \setminus E(H)$. Bridges B_3 and B_6 are trivial bridges. Bridges B_1 and B_2 are equivalent 3-bridges since they both have the same three vertices of attachment. B_1 has one internal vertex and B_2 has four internal vertices.

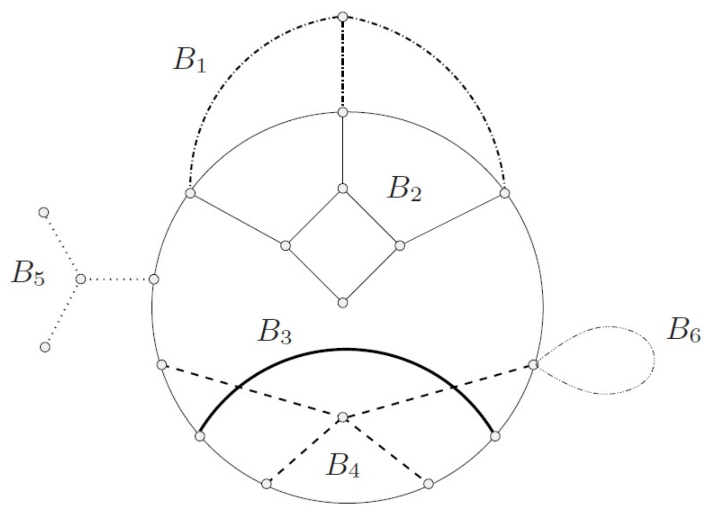


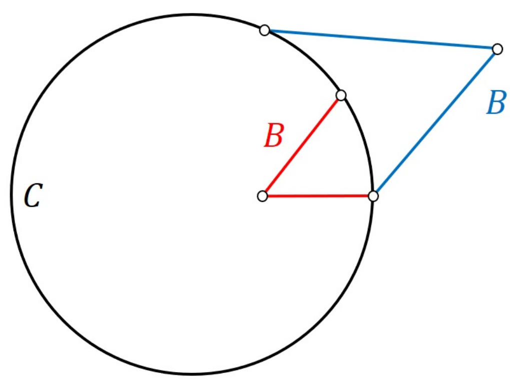
Figure 10.15

Note. We now concentrate on bridges of a cycle. The vertices of attachment on the cycle can be shared between bridges or not, or can be interwoven in a sense. These ideas lead to the following definitions.

Definition. The vertices of a k-bridge B of a cycle C in a connected graph G, where $k \geq 2$, yield a (edge) partition of C into k edge-disjoint paths (the paths joining "consecutive" vertices of attachment of the bridge), called the *segments* of B. Two bridges (each with at least two vertices of attachment) of C *avoid* each other if all vertices of attachment of one bridge lie in a single segment of the other bridge; otherwise the two bridges *overlap*. Two bridges B and B' are *skew* if there are distinct vertices of attachment u and v of B, and u' and v' of B' which occurs in the cyclic order u, u', v, v' in C.

Note. In Figure 10.15, bridges B_2 and B_3 avoid each other. Bridges B_1 and B_2 overlap (and they share the same vertices of attachment), and bridges B_3 and B_4 overlap (and they share no vertices of attachment). Bridges B_3 and B_4 are skew; bridges B_1 and B_2 are not skew.

Note. "Avoid" is a slightly misleading term since bridges B and B' can avoid each other and yet share vertices of attachment:



Theorem 10.25. Overlapping bridges of a cycle in a connected graph are either skew or else equivalent 3-bridges.

Note. We now consider bridges of cycles is connected plane graphs. Since a cycle in a plane graph (with the embedding given) is a simple closed curve, so by the Jordan Curve Theorem (Theorem 10.1), every bridge of cycle C in G is contained in either int(C) or in ext(C).

Definition. Let G be a connected plane graph containing cycle C. Those bridges of C contained in $\mathrm{int}(C)$ is an *inner bridge*, and a bridge of C contained in $\mathrm{ext}(C)$ is an *outer bridge*.

Note. In Figure 10.16, bridges B_1 and B_2 of the cycle are inner bridges of the given plane graph, and bridges B_3 and B_4 of the cycle are outer bridges. The next result concerning inner (and outer) bridges is not surprising, when we consider the geometry of the claim.

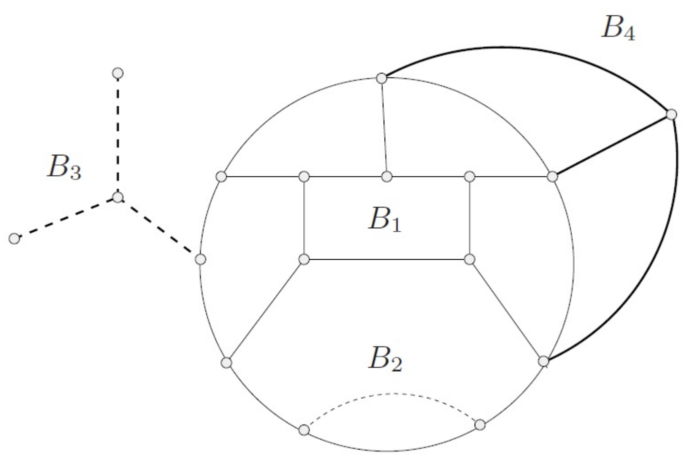


Figure 10.16

Theorem 10.26. Let G be a plane graph containing cycle C. The inner bridges of C avoid one another, and the outer bridges of C avoid one another.

Definition. Let G be a connected graph and C a cycle in G. The *bridge-overlap graph* of C is the graph whose vertex set is the set of all bridges of C in G, with two bridges in the bridge-overlap graph being adjacent if the bridges overlap.

Note. By Theorem 10.26, we see that the bridge-overlap graph of a cycle C in a plane connected graph G is bipartite with one partite set consisting of the bridges in $\text{int}(C)$ and the other partite set consisting of bridges in $\text{ext}(C)$. Notice that "bridge," "overlap," and "skew" are defined for any graph, but "inner bridge" and "outer bridge" are only defined for plane graphs. So the condition of the bridge-overlap graph being bipartite is a necessary condition for graph G being planar. In Exercise 10.5.7 it is to be shown that the converse holds; that is, if the bridge-overlap graph is bipartite for each cycle C in connected graph G, then G is a planar graph.

Note. Recall that a given planar graph may have different planar embeddings. For example, in Figure 10.11 we say two embeddings of a graph that have faces of different degrees (so that the duals of the embeddings are not isomorphic). We now consider what it means for a planar graph to have "only one" planar embedding and give a class of graphs which have unique planar embeddings.

Definition. Two planar embeddings of a plane graph are *equivalent* if their face boundaries (taken as sets of edges) are identical. A planar graph for which any two planar embeddings are equivalent are said to have a *unique embedding* in the plane. A cycle C in a connected graph G is *nonseparating* if it has no chords and at most one nontrivial bridge.

Note. Let G be a loopless graph which is not itself a cycle. Cycle C has no chords if and only if it is an (vertex) induced subgraph of G. Since G is loopless, then cycle C has at most one nontrivial bridge (loops and chords at most one nontrivial bridge (loops and chords would be trivial bridges) if and only if $G - V(C)$ is connected (it is a connected component F of $G - V(C)$ as described in the first class of bridges). Therefore, a cycle in G is nonseparating if and only if it is an induced subgraph of G and $G - V(C)$ is connected.

Definition. For plane graph G, a cycle in G which is a boundary of a face is a *facial cycle*.

Note. W. T. Tutte proved in 1963 that in a simple 3-connected plane graph, facial and nonseparating cycles are the same, as given in the next theorem. William Thomas Tutte (May 17, 1917 – May 2, 2002) is a prominent member of the graph theory community. He published 168 papers and several books; he was active from around the beginning of the second world war until the late 1990s. Bondy and Murty's book is dedicated, in part, to Tutte.

BILL TUTTE

Theorem 10.27. A cycle in a simple 3-connected plane graph is a facial cycle if and only if it is nonseparating.

Note. The following was proved by H. Whitney in 1933, but the proof given here is based on Tutte's 1963 Theorem 10.27.

Theorem 10.28. Every simple 3-connected planar graph has a unique planar embedding.

Note. Since a planar embedding determines a dual of a plane graph, we have the following.

Corollary 10.29. Every simple 3-connected plane graph has a unique dual graph.

Revised: 9/7/2020