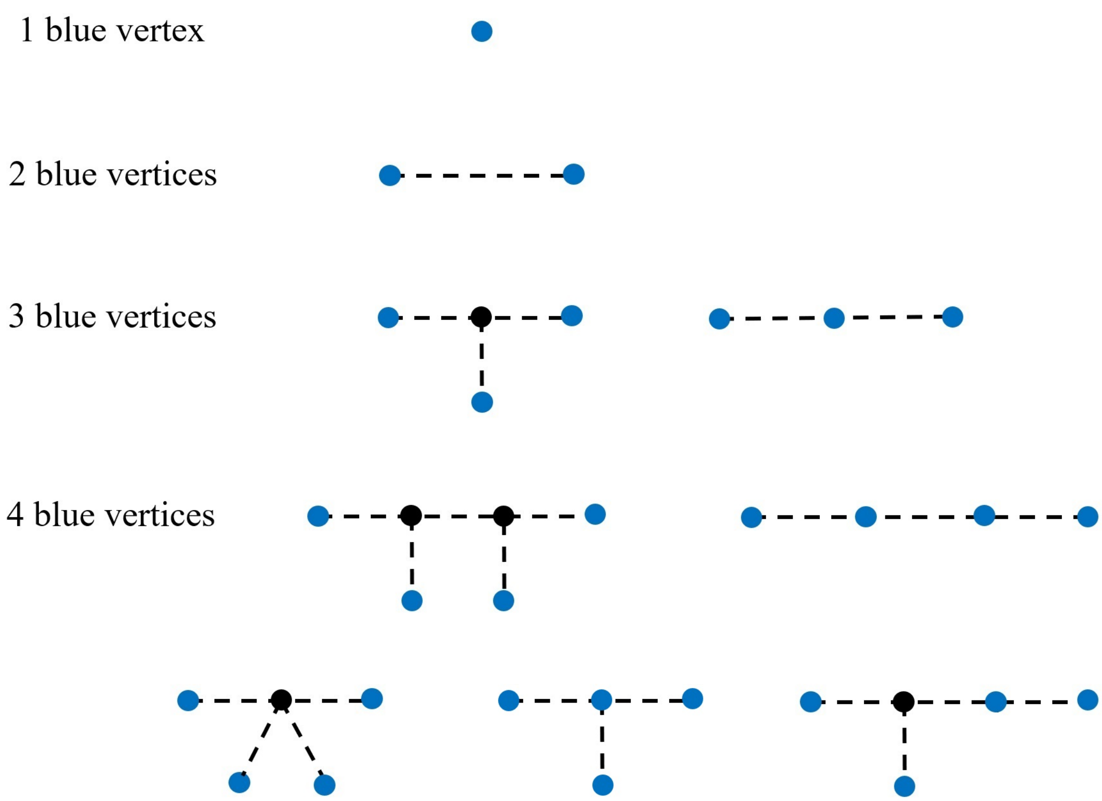
Supplement. Section 10.5. Kuratowski's Theorem, Exercise 10.5.3(b) "Any graph which has a K_5-minor contains a Kuratowski subdivision."

Note. This supplement includes a proof that: "Any graph which has a K_5-minor contains a Kuratowski subdivision." This then gives that Wagner's Theorem (Theorem 10.32) implies Kuratowski's Theorem (Theorem 10.30). The proof given here seems lengthy and inelegant! Constructive criticism/suggestions are welcome.

Exercise 10.3.3(b) Any graph which has a K_5-minor contains a Kuratowski subdivision.

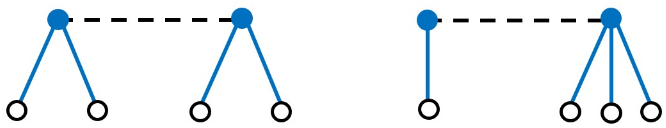
Proof. Since G has a K_5-minor, by Note 10.5.A there is a partitioning of the vertex set of G into $V_0, V_1, V_2, V_3, V_4, V_5$ where the subgraphs $G[V_i]$ are connected for $1 \leq i \leq 5$ and the vertices of set V_i are associated with vertex v_i of K_5 for $1 \leq i \leq 5$. Since each vertex of K_5 is adjacent to each other vertex of K_5, then for $i \neq j$ there is a vertex $v_i \in V_i$ and $v_j \in V_j$ such that v_i and v_j are adjacent in G. So each V_i contains 1, 2, 3, or 4 such vertices, which we call "blue vertices" (since this is the color we use to represent them in figures below). Since $G[V_i]$ is connected, then there is a spanning tree T of $G[V_i]$ by Proposition 4.6. If V_i contains two blue vertices, then there is a unique path in T between these two blue vertices by Proposition 4.1. If there is a third blue vertex in V_i then there is a shortest path in T from this third vertex to the path joining the first two vertices (for each vertex of the first path, find the unique path joining this vertex to the third blue vertex, then choose one such path of shortest length). Add this path to the first path, giving a tree containing the three blue vertices. Similarly, if there is a fourth blue vertex in V_i then find a shortest path in T between it and the constructed tree containing the first three blue vertices and add this to the constructed tree to produce a tree containing all four blue vertices. Now in graph G, the blue vertices are, together, adjacent to four other vertices because they are used to give the edges incident to the vertex v_i of the K_5-minor that results when identifying the vertices of V_i.

We now consider the possible configurations of the constructed trees. We denote by a dashed line segment an edge that may be subdivided several times (thus giving a path). Based on the number of blue vertices in V_i, we have the following configurations:

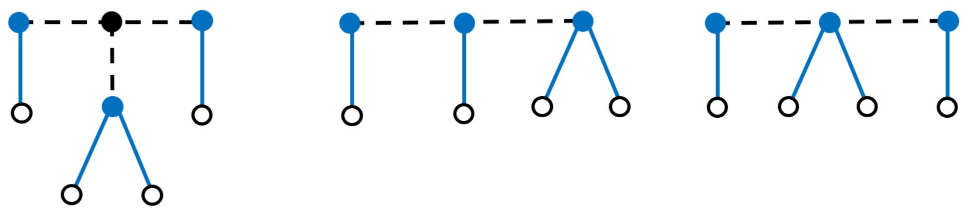


We next add four edges (associated with the edges of minor K_5). There is only one way to do this when there is one or four blue vertices in the configuration, but there may be more than one way when the configuration has two or three blue vertices. For configurations with two or three blue vertices we then have (representing the four edges in blue, and not repeating symmetric cases):

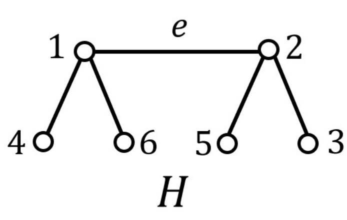
2 blue vertices



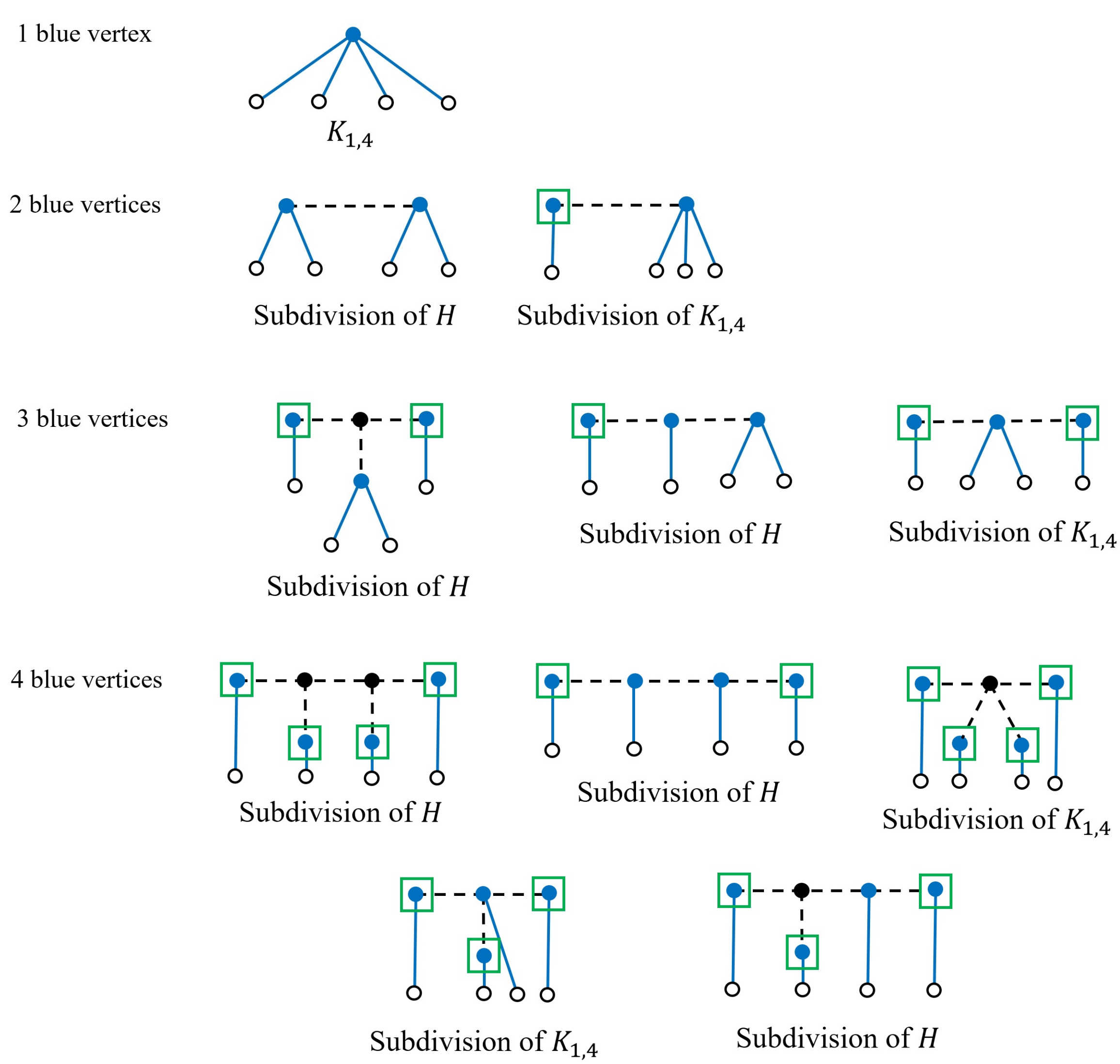
3 blue vertices



Next, we show that each of the configurations including the blue vertices, along with the four blue edges, are either subdivisions of $K_{1,4}$ or subdivisions of graph H where H is the graph:



In the following, some of the vertices of the configuration have a square around them. If one of the edges incident to such a vertex is contracted (that is, if we "undo" edge subdivision by this vertex) then it can be seen that either a subdivision of $K_{1,4}$ or a subdivision of graph H results:



If V_i leads to a configuration that is a subdivision of H, then a sequence of edge and vertex deletions and edge contractions gives the graph G_1 of Note 10.5.C (the graphs $G[V_j]$ are collapsed to single vertices for $j \neq i$, $j \neq 0$). As shown in Note 10.5.C, graph G has a $K_{3,3}$ subgraph, so that $K_{3,3}$ is a minor of G in this case. Then, by Exercise 10.5.3(a), G contains a $K_{3,3}$-subdivision, as claimed.

The leaves the case where each V_i, $1 \leq i \leq 5$, leads to a configuration which is a subdivision of $K_{1,4}$. In this case, the five subdivisions of $K_{1,4}$ form a subdivision of K_5 in G, as claimed. ∎

Revised: 3/24/2023