Section 10.5. Kuratowski's Theorem

Note. We saw in Section 10.3, "Euler's Formula," that K_5 and $K_{3,3}$ are nonplaner (see Corollaries 10.23 and 10.24). In 1930, Kazimierz (aka. "Casimir") Kuratowski gave the classification of nonplanar graphs in terms of subgraphs related to K_5 and $K_{3,3}$ in "Sur le poblème des courber gauches en topologie," *Fundamenta Mathematicae*, **15**, 271–283. A copy is available online in French (accessed 6/5/2020).

Kazimierz Kuratowski (February 2, 1896 - 18 June 18, 1980)

Photo from MacTutor History of Mathematics Archive (accessed 6/5/2020).

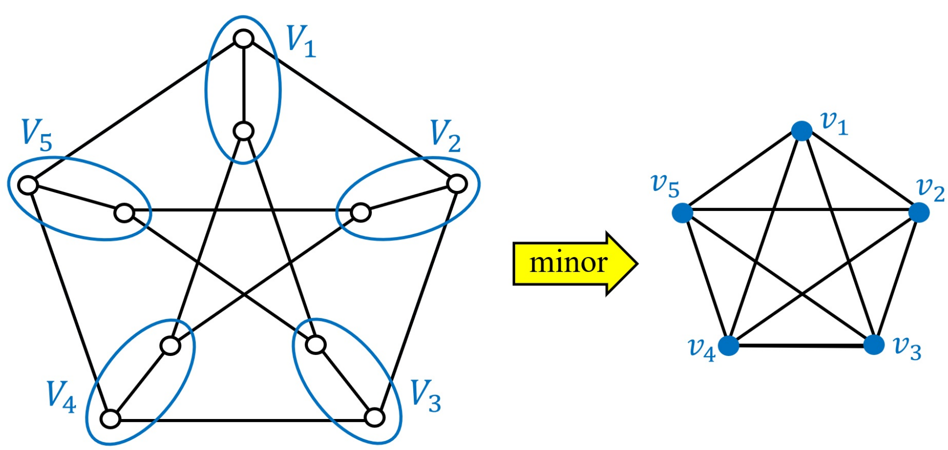
Theorem 10.30. Kuratowski's Theorem.
A graph is planar if and only if it contains no subdivision of either K_5 or $K_{3,3}$.

Note. We introduce the idea of a graph minor and present a proof by Carsten Thomassen from "Kuratowski's Theorem," *Journal of Graph Theory*, **5**(3), 225–241 (1981).

Definition. A *minor* of a graph G is any graph obtainable from G through a sequence of vertex and edge deletions and edge contractions. If F is a minor of G, we write $F \preceq G$. An F*-minor* of G is a minor of G which is isomorphic to F. A minor which is isomorphic to K_5 or $K_{3,3}$ is a *Kuratowski minor*. A subdivision of K_5 or $K_{3,3}$ is a *Kuratowski subdivision*.

Note 10.5.A. We can think of constructing minors as follows. Take a partition $(V_0, V_1, \ldots, V_k)$ of the vertex set V of graph G such that the induced subgraphs $G[V_i]$ are connected for $1 \le i \le k$. Let H be the graph obtained from G by deleting V_0 and shrinking each induced subgraph $G[V_i]$ to a single vertex (that is, contract all edges of $G[V_i]$ and then identify all resulting vertices in V_i; see Section 2.3. Modifying Graphs). Then any spanning subgraph F of H is a minor of G.

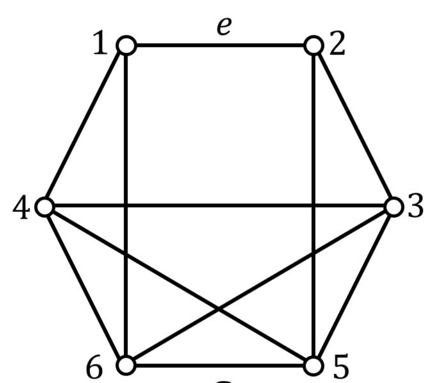
Note. K_5 is a minor of the Petersen graph, as shown in the following figure:



Notice that the Peterson graph does not contain a K_5-subdivision, since this would require it to contain five vertices, each of degree four, but it is 3-regular.

Note 10.5.B. A graph which contains an F-subdivision also has an F-minor. To obtain F as a minor, we delete the vertices and edges not in the F-subdivision and then contract each subdivided edge into a single edge. By Exercise 10.5.3(a), the converse holds if F has maximum degree three or less. That is, a graph with an F-minor for such F also contains an F-subdivision.

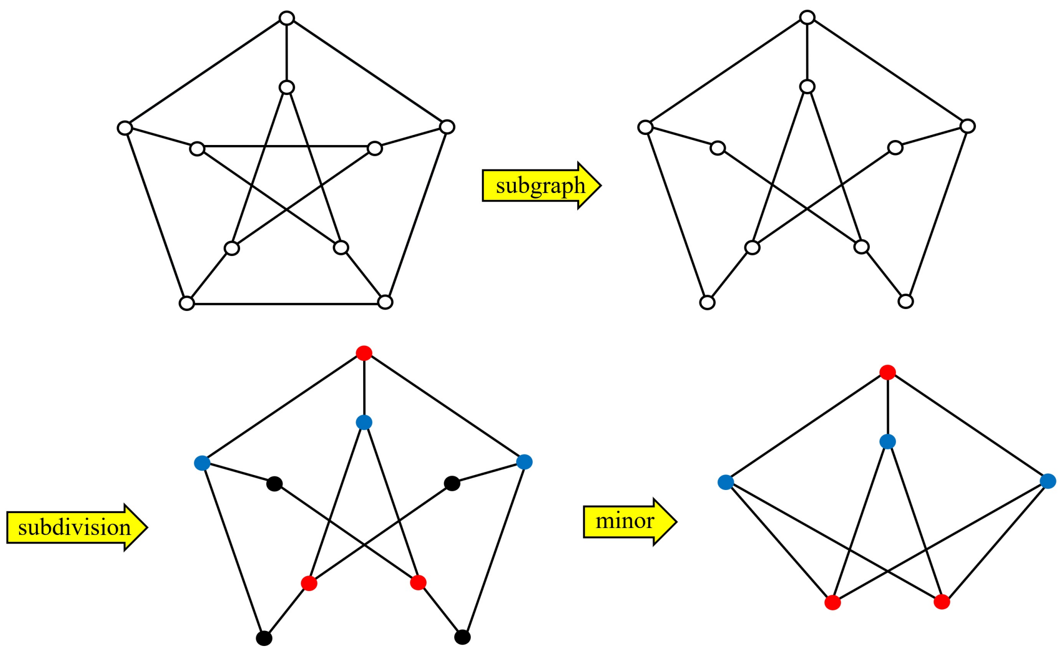
Note 10.5.C. To see that a graph may have an F-minor but not an F-subdivision, consider the graph G_1 below.



G_1

Then G_1 has a K_5-minor (notice the contraction of edge e produces $G_1/e \cong K_5$), but G_1 does not contain a subdivision of K_5 (this would require five vertices of degree 4). However, G_1 does contain a $K_{3,3}$-subdivision (in fact, $K_{3,3}$ is a subgraph of G_1). Deleting edges 35 and 46 gives $K_{3,3}$ with partite sets $\{1, 3, 5\}$ and $\{2, 4, 6\}$. The Petersen graph behaves similarly in that it has a K_5-minor graph but no K_5-subdivision; it has a $K_{3,3}$-subdivision but not a $K_{3,3}$ subgraph (as we show below). In Exercise 10.5.3(b) it is to be shown that a graph with a K_5-minor contains either a K_5-subdivision or a $K_{3,3}$-subdivision.

Note. As an additional example of minors, subdivisions, and subgraphs, consider the Petersen graph again. In the figure below, we first delete two edges of the Petersen graph, as shown (upper right). This results in a subdivision of $K_{3,3}$ as shown (lower left) where the partite sets are the three red vertices and the three blue vertices. The three black vertices represent subdivisions of edges of $K_{3,3}$. If we contract one edge incident to each black vertex, then we get the graph $K_{3,3}$ (lower right). Therefore $K_{3,3}$ is a both a subdivision and a minor of the Petersen graph (consistent with Note 10.5.B).



We saw above that the Petersen graph has a K_5 minor, but it does not have a K_5 subgraph (since the Petersen graph is 3-regular and K_5 if 4-regular). We also claim that the Petersen graph does not have a $K_{3,3}$ subgraph. *If* there is a $K_{3,3}$ subgraph, then consider an arbitrary vertex (we need to use the fact that the Petersen graph is vertex transitive here, which it is), say the upper-most vertex in the drawing of the Petersen graph given above, and let it be in one partite set X of the bipartition.

Then each of the three neighbors of this vertex must be in the other partite set Y of the bipartition. But then each of the three vertices in Y must share two more common neighbors, and this is not the case. So there can be no $K_{3,3}$ subgraph of the Petersen graph.

Note. By Note 10.2.C, the deletion or contraction of an edge in a planar graph results in a planar graph. Since a minor graph is found through a sequence of edge deletions and contractions, then we have the following.

Proposition 10.31. Minors of planar graphs are planar.

Note. We will argue that Kuratowski's Theorem is equivalent to the following result of K. Wagner from 1937, which we will prove.

Theorem 10.32. WAGNER'S THEOREM.
A graph is planar if and only if it has no Kuratowski minor.

Note 10.5.D. By Note 10.5.B, a graph that contains an F-subdivision also has an F-minor, so Kuratowski's Theorem (Theorem 10.30) implies Wagner's Theorem (Theorem 10.32). Since $K_{3,3}$ has maximum degree three, then by Exercise 10.5.3(a) any graph with a $K_{3,3}$-minor also contains a $K_{3,3}$-subdivision. By Exercise 10.5.3(b), any graph which has a K_5-minor necessarily contains a Kuratowski subdivision; see Supplement. Section 10.5. Kuratowski's Theorem, Exercise 10.5.3(b) for a solution to this exercise. So Wagner's Theorem implies Kuratowski's Theorem. That is, Kuratowski's Theorem and Wagner's Theorem are equivalent. We will prove Wagner's Theorem using the next two lemmas.

Note. Recall from Section 9.1. Vertex Connectivity, a k-vertex cut of G is a set S of k vertices of G such that for some vertices u and v of G, u and v are in different components of $G - S$. Set S is called a *uv-vertex cut*. In Section 9.4. Three-Connected Graphs, the components of $G - S$ are called *S-components*. If G is connected and $S = \{x, y\}$ then the new edge xy is a marker edge and the S-components of G with edge xy added are the marked S-components. See Figure 9.7.

Lemma 10.33. Let G be a graph with a 2-vertex cut set $S = \{x, y\}$. Then each marked $\{x, y\}$-component of G is isomorphic to a minor of G.

Lemma 10.34. Let G be a graph with a 2-vertex cut set $S = \{x, y\}$. Then G is planar if and only if each of its marked S-components is planar.

Note. We next prove Wagner's Theorem (Theorem 10.32) for 3-connected graphs. The proof is based on C. Thomassen's 1981 paper mentioned above. It uses Bondy and Murty's Theorem 9.10 (also proved by Thomassen) which we recall here.

> **Theorem 9.10.** Let G be a 3-connected graph on at least five vertices. Then G contains an edge e such that G/e is 3-connected.

Theorem 10.35. Every 3-connected nonplanar graph has a Kuratowski minor.

Definition. A *convex embedding* is a planar embedding all of whose faces are bounded by convex polygons.

Note. Similar to the proof of Theorem 10.35, we can prove that every simple 3-connected graph has a convex embedding. This is to be shown in Exercise 10.5.5 (the proof is similar to the proof of Theorem 10.35, but slightly modified in the placing of bridges B_x and B_y and edge $e = xy$).

Note. We are now ready to give a proof of Wagner's Theorem (Theorem 10.32), and hence a proof of Kuratowski's Theorem (Theorem 10.30) by Note 10.5.D.

Note. Three of the exercises in this section give other classifications of planar graphs. The proof of each is based on Kuratowski's Theorem (Theorem 10.30). We paraphrase these exercises here.

Exercise 10.5.7. A graph is planar if and only if the bridge-overlap (see Section 10.4. Bridges) of each cycle is bipartite.

Exercise 10.5.8. A basis of the cycle space (see Section 2.6. Even Subgraphs) of a graph is a 2-*basis* if each member of the basis is a cycle of the graph, and each edge of the graph lies in at most two of these cycles. Prove that a graph is planar if and only if its cycle space has a 2-basis. This is "MacLane's Theorem" and first appeared in Saunders MacLane, A Combinatorial Condition for Planar Graphs, *Fundamenta Mathematicae*, **28**, 22–32 (1937). An online copy is available at the Fundamenta Mathematicae website (accessed 1/13/2021).

Exercise 10.5.9. A graph H is called an *algebraic dual* of a graph G if there is a bijection $\varphi : E(G) \to E(H)$ such that a subset C of $E(G)$ is a cycle of G if and only if $\varphi(C)$ is a bond of H (see Exercise 4.3.8). Prove that a graph is planar if and only if it has an algebraic dual. This is "Whitney's Theorem" and first appeared in Hassler Whitney, Non-Separable and Planar Graphs, *Transactions of the American Mathematical Society*, **34**, 339–362 (1932). An online copy is available at the T.A.M.S. website (accessed 1/13/2021).

Note. In the next section we will consider embedding graphs on surfaces. A generalization of Kuratowksi's Theorem holds in this setting also. For a surface S, there is a finite collection $\mathbf{Forb}_0(S)$ of "forbidden" graphs that cannot be drawn on S such that any graph G can be drawn on S if and only if G has no graph in $\mathbf{Forb}_0(S)$ as a minor. This was proved in Neil Robertson and P.D. Seymour, Graph Minors. VIII. A Kuratowski Theorem for General Surfaces, *Journal of Combinatorial Theory Series B*, **48**, 255–288 (1990). A copy is available online at J.C.T.B. website (accessed 1/13/2021). More references and related results can be found in Bojan Mohar and Carsten Thomassen, *Graphs on Surfaces*, Baltimore: Johns Hopkins University Press (2001); see its Chapter 6 on "Embedding Extensions and Obstructions."

Note. At the end of this section, Bondy and Murty describe an algorithm that has as input a 3-connected graph G on four or more vertices and has as output either a Kuratowski minor (if G is nonplanar) or a planar embedding of G (when G is planar). As described in the proof of Wagner's Theorem (Theorem 10.32), it suffices to consider 3-connected graphs. The algorithm, "Algorithm 10.36. Planarity Recognition and Embedding," runs in polynomial-time. Bondy and Murty describe the algorithm as (see page 272–73):

> "First, the input graph is contracted, one edge at a time, to a complete graph on four vertices (perhaps with loops and multiple edges) in such a way that all intermediate graphs are 3-connected. . . . The resulting four-vertex graph is then embedded in the plane. The contracted edges are now expanded one by one (in reverse order). At each stage of this expansion phase, one of two eventualities may arise: either the edge can be expanded while preserving planarity, and the algorithm proceeds to the next contracted edge, or else two bridges are found which overlap, yielding a Kuratowski minor."

If the graph is nonplanar then the algorithm outputs the Kuratowksi minor thus "certifying" that the input graph is nonplanar. If the graph is planar, then the algorirhm outputs a planar embedding of G.

Algorithm 10.36. Planarity Recognition and Embedding.

Input: a 3-connected graph G on four or more vertices

Output: a Kuratowksi minor of G or a planar embedding of G

1. *set* $i := 0$ *and* $G_0 := G$

Contraction Phase:

2. **while** $i < n - 4$ **do**

3. *find a link* $e_i := x_iy_i$ *of* G_i *such that* G_i/e_i *is 3-connected*

4. *set* $G_{i+1} := G_i/e_i$

5. *replace* i *by* $i + 1$

6. **end while**

EXPANSION PHASE:

7. *find a planar embedding* $\tilde{G}_{n-4}$ *of the four-vertex graph* G_{n-4}

8. *set* $i := n - 4$

9. **while** $i > 0$ **do**

10. *Let* C_i *be the facial cycle of* $\tilde{G}_i - z_i$ *that includes all the neighbors of* z_i *in* $\tilde{G}_i$*, where* z_i *denotes the vertex of* $\tilde{G}_i$ *resulting from the contraction of the edge* e_{i-1} *of* G_{i-1}

11. *let* B_i *and* $\tilde{B}_i$*, respectively, denote the bridges of* C_i *containing the vertices* x_{i-1} *and* $y_{i=1}$ *in the graph obtained from* G_{i-1} *by deleting* e_{i-1} *and all other edges linking* x_{i-1} *and* y_{i-1}

12. **if** B_i *and* B'_i *are skew* **then**

13. *find a* $K_{3,3}$*-minor* K *of* G_{i-1}

14. *return* K

15. **end if**

16. **if** B_i *and* B'_i *are equivalent 3-bridges* **then**

17. *find a* K_5*-minor* K *of* G_{i-1}

18. *return* K

19. **end if**

20. **if** B_i *and* B'_i *avoid each other* **then**

21. *extend the planar embedding* $\tilde{G}_i$ *of* G_i *to a planar embedding* $\tilde{G}_{i-1}$ *of* G_{i-1}

22. *replace* i *by* $i - 1$

23. **end if**

24. **end while**

25. return $\tilde{G}_0$.

Note. The Planarity Recognition and Embedding algorithm runs in polynomial time, so that it belongs to the class of polynomial-time algorithms, $\mathcal{P}$; see my notes for Mathematical Modeling Using Graph Theory (MATH 5870) on Section 8.1. Computational Complexity (see Note 8.1.A). J. Hopcroft and R. Tarjan, in "Efficient Planarity Testing," *Journal of the Association for Computing Machinery*, **21**(4), 549–568 (1974) (a copy is online on the Princeton University Computer Science server; accessed 1/13/2023) gave a linear time(!) planarity algorithm. A graph planarity algorithm using bridge-overlap is given in Bondy and Murty's undergraduate-graduate level *Graph Theory with Applications* (Macmillan Press Ltd., 1976); see Section 9.8, "Planarity Algorithm."

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