Section 10.6. Surface Embeddings of Graphs

Note. We now take the idea of planar embeddings and extend it to embeddings on surfaces other than the plane. We will informally classify surfaces, both orientable and nonorientable. We give a version of Euler's Formula for surfaces and state the Orientable Embedding Conjecture. This section lacks rigor and we give no proofs.

Note. An n-manifold is sort of an n-dimensional surface which has a certain level of smoothness. The formal definition is rather involved and best dealt with in a class on differential geometry. A formal definition can be found in my Differential Geometry (MATH 5310) online notes on VII. Manifolds (see Definition VII.2.01). Another formal definition is in my online notes for Complex Analysis 2 (MATH 5520) on IX.6. Analytic Manifolds (see Definition IX.6.2) where analytic functions from $\mathbb{C}$ to $\mathbb{C}$ are part of the definition. A somewhat more tangible definition of a 2-manifold (which is sufficient for our needs here in graph theory; we don't need the full blown definition of an n-manifold for $n \neq 2$) is given in another set of Differential Geometry (MATH 5310) notes on 1.9. Manifolds. A less rigorous, informal PowerPoint presentation with several animations illustrating what it looks like to "live" in various 2-manifolds is given in my online talk on The Big Bang and the Shape of Space.

Definition. A *surface* is a connected 2-dimensional manifold. The *cylinder* may be obtained by gluing together two opposite sides of a rectangle. The *Möbius band* may be obtained by gluing together two opposite sides of a rectangle after making one half-twist. The *torus* may be obtained by gluing together the two open ends of a cylinder. The *Klein bottle* may be obtained by gluing together the two open ends of a cylinder after making one half-twist.

Note. A planes, spheres, cylinders, Möbius band, torus, and Klein bottle are each examples of surfaces. Figure 10.24 gives pictures of a Möbius band and a torus.

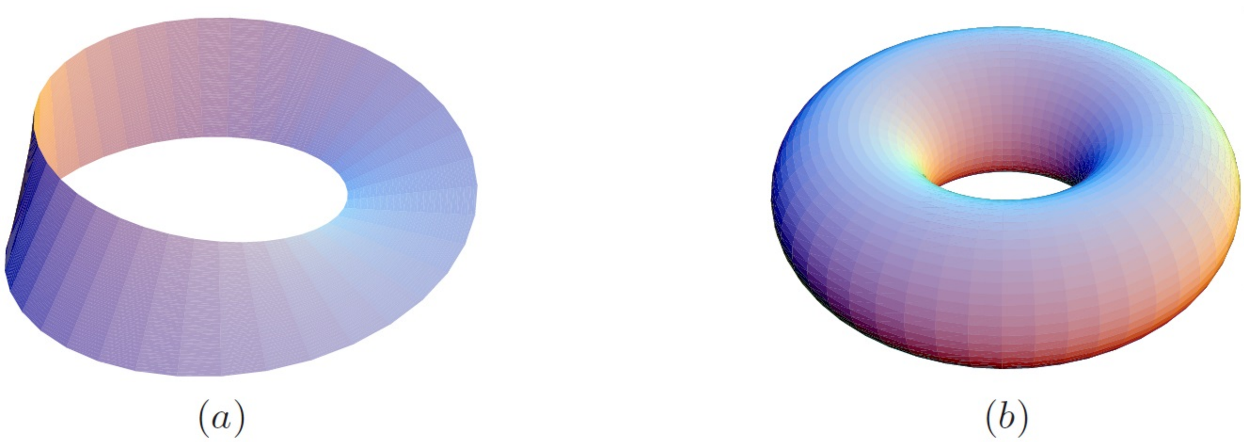
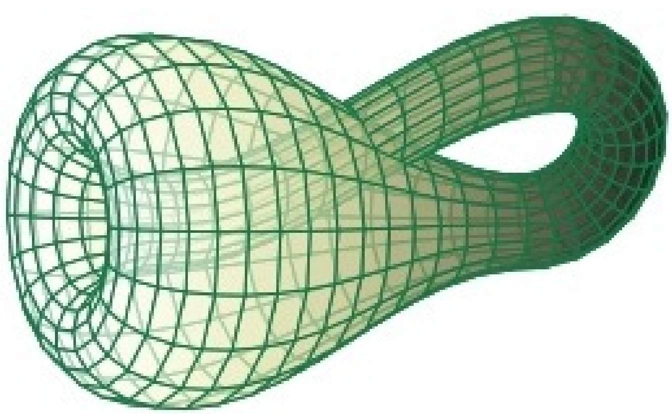


Figure 10.24. (a) The Möbius band and (b) the torus.

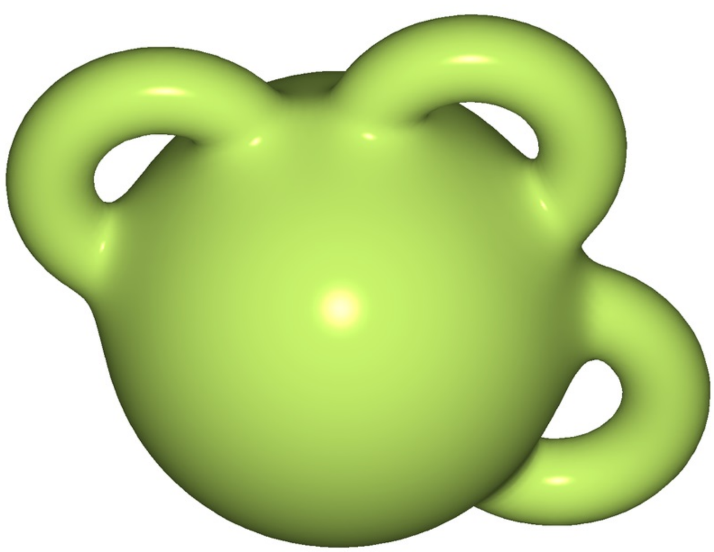
A Klein bottle cannot be embedded in $\mathbb{R}^3$ without penetrating the walls of the surface (however, the Klein bottle has the properties of a surface; it *can* be embedded in $\mathbb{R}^4$ without penetrating walls, but this is not really relevant to the Klein bottle as a surface). The following image is from the Gnuplotting software website (accessed 1/14/2021).



Note/Definition. We now break surfaces into two types. Consider an inhabitant of the Möbius band. If the inhabitant starts at a particular point, goes "around" the Möbius band, and returns to the starting point then the inhabitant will find that items in the space that were on its left are now on its right and vice versa. Another inhabitant of the Möbius band which watches this movement will see the traveler returned "reversed" with the traveler having his left and right sides interchanged. My favorite way to illustrate this is with a winking smiley emoji, since the wink lets us distinguish left from right (this is how I illustrate these ideas in my PowerPoint presentation on The Big Bang and the Shape of Space). Bondy and Murty describe this in terms of an ant that crawls *on* the Möbius band, but in these discussions we need to consider inhabitants that are *in* the surface, not on it. A surface with this type of reversing property is a *nonorientable* surface. A surface without this reversing property is an *orientable* surface. The plane, the sphere, the cylinder, and the torus are orientable surfaces. The Möbius band, the Klein bottle, and the "projective plane" (which is a surface formed from a rectangle by gluing together two opposite sides of a rectangle after making one half-twist and then by gluing together the other two opposite sides after making one half-twist; this should not be confused with the finite projective plane mentioned in Exercise 1.3.13) are nonorientable surfaces.

Note/Definition. A surface is *closed* if it is bounded and has no boundary. The Möbius band has a boundary which is *homeomorphic* (that is, continuously deformable) to a circle, so it is not a closed surface. The plane is not bounded and so is not a closed surface. The simplest closed surface is the sphere. The "next simplest" is the torus. The closed surfaces other than the sphere are *higher surfaces.*

Note. All higher surfaces can be constructed by starting with a sphere and performing two operations. We first give Bondy and Murty's outline, then we'll give some other relevant references. Let S be a sphere, let D_1 and D_2 be two disjoint discs of equal radii on S, and let H be a cylinder of the same radius as D_1 and D_2. The operation of *adding a handle* to S at D_1 and D_2 consists of cutting out D_1 and D_2 from S and then bending and attaching H to S is such a way that the rim of one of the ends of H coincides with the boundary of D_1 and the rim of the other end of H coincides with the boundary D_2. A *sphere with k handles* is the surface obtained from a sphere by adding k handles, denoted S_k. Surface S_k is of *genus* k. The following figure (from the Wikipedia page on Handle Decomposition, accessed 1/14/2021) is of a sphere with 3 handles, S_3:



The torus is homeomorphic to a sphere with one handle and the double torus (a "fat figure 8"; see also Exercise 10.6.2) is homeomorphic to a sphere with two handles.

Every closed orientable surface is homeomorphic to a sphere with k handles for some $k \geq 0$.

Note. Now for nonorientable closed surfaces. Let S be a sphere, let D be a disk on S, and let B be a Möbius band whose boundary has the same length as the circumference of D. The operation of *adding a cross-cap to S at D* consists of attaching B to S so that the boundaries of D and B coincide. The surface obtained from the sphere by attaching one cross-cap is the *projective plane* and is the simplest nonorientable closed surface. A sphere with k cross-caps is denoted by N_k, the index k being its *cross-cap number*. Every closed nonorientable surface is homeomorphic to N_k for some $k \geq 1$.

Note. Combining the above claims, we have:

> **The Classification Theorem for Closed Surfaces.**
>
> Every closed surface is homeomorphic to either S_k or N_k, for a suitable value of k.

We can add both handles and cross-caps to a sphere. The surface obtained from the sphere by adding $k > 0$ handles and $\ell > 0$ cross-caps is homeomorphic to $N_{2k+\ell}$.

Note. In 1992, John H. Conway of Princeton University gave a completely new proof of the Classification Theorem. He called it his "Zero Irrelevancy Proof" (hence, "ZIP"). A very readable and well-illustrated version of the proof is given in: George Francis and Jeffrey Weeks, Conway's ZIP Proof, *American Mathematical Monthly*, **106**, 293–399 (1999). A copy is available on Andrew Ranicki's webpage (accessed 1/14/2021). This is also reprinted as Appendix C in Jeffrey Week's *The Shape of Space*, Second Edition, Basel: Marcel Dekker (2002). A more traditional proof from a class on algebraic topology is given in James Munkres, *Topology*, Second Edition, Upper Saddle River, NJ: Prentice Hall (2000). We give the statement of Munkres result here

> **Theorem 77.5. The Classification Theorem.**
>
> Let X be the quotient space obtained from a polygonal region in the plane by pasting its edges together in pairs. Then X is homeomorphic either to S^2 [a sphere], to the n-fold torus T_n [a sphere with n handles, S_n], or to the m-fold projective plane P_m [a sphere with k cross-caps, N_m].

As this version of The Classification Theorem suggests, every closed surface results from identifying ("gluing") the sides of some polygon (in fact, this is how we defined the torus and Klein bottle above). In the terminology of Week's *Shape of Space*, the polygon is the *fundamental domain* of the surface and the identifying of sides is related to a type of "modding out" (as is done in a "quotient space," similar to the behavior of a quotient group determined by a normal subgroup in the group theoretic setting). This can be used to illustrate how to embed certain graphs on surfaces (at least for not-too-complicated surfaces). Figure 10.25 gives embeddings on the projective plane of K_6 and the Petersen graph. We might need to explain the context some more. The projective plane can be created by identifying points on the boundary of a disk which are opposite each other. So in Figure 10.25(a), the six vertices on the outer circle actually only represent three vertices (the top and bottom vertices are identified, for example). Similarly, in Figure 10.25(b), the edges that "disappear" on the dotted circle, "reappear" on the opposite side of the circle.

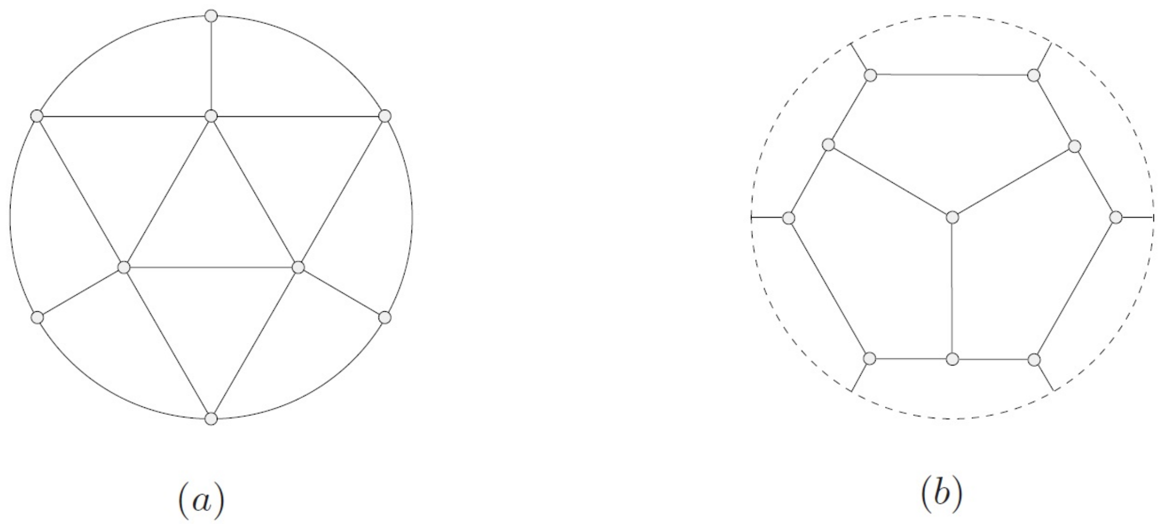


Figure 10.25. Embeddings on the projective plane of (a) K_6 and (b) the Petersen graph.

Definition. An embedding $\tilde{G}$ of a graph G on a surface Σ is a *cellular embedding* if each of the arcwise-connected regions of $\Sigma \setminus \tilde{G}$ is homeomorphic to the open disk. These regions are the *faces* of $\tilde{G}$, and their number is denoted $f(\tilde{G})$.

Note. Bondy and Murty comment that "...all the embeddings that we discuss are assumed to be cellular." At least, this is the case for the following theorem and corollaries of this section. The embedding of K_4 on the torus in Figure 10.26(a) is a cellular embedding, but the embedding on the torus in Figure 10.26(b) is not cellular.

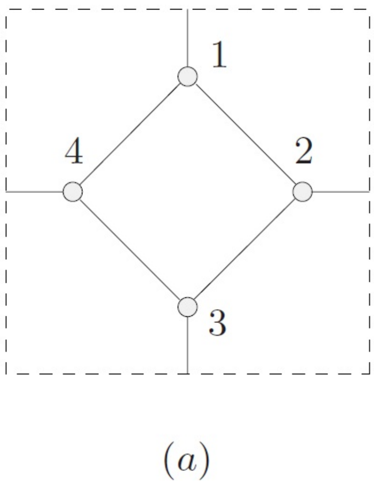
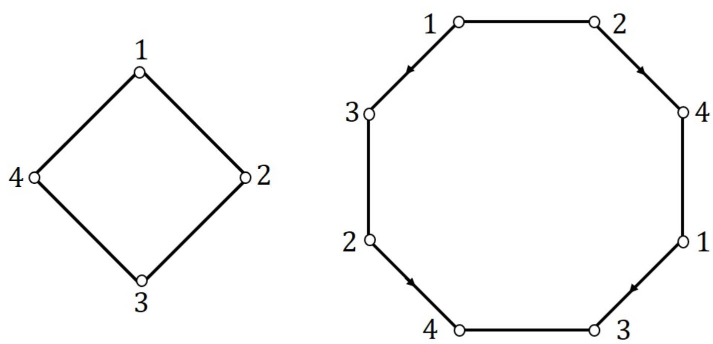
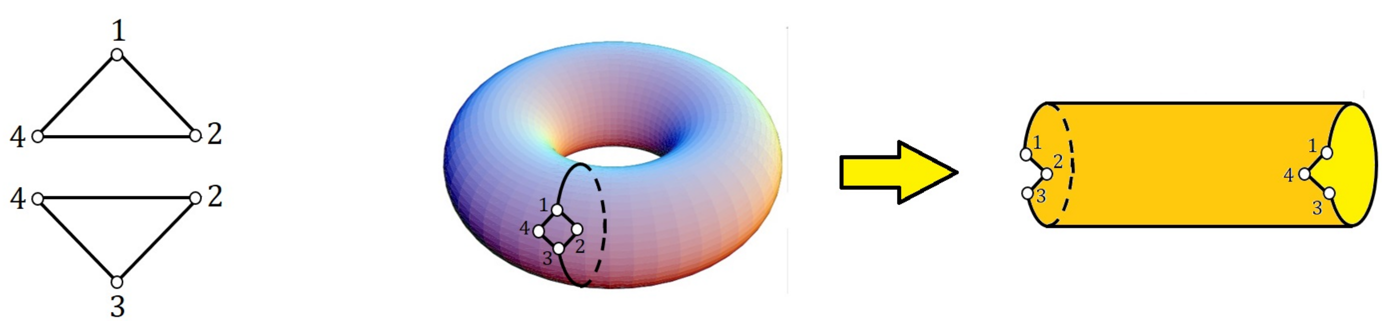


Figure 10.26. Two embeddings of K_4 on the torus: (a) a cellular embedding, and (b) a noncellular embedding.

In Figure 10.26(a), the walks 12341 and 124134231 produce the faces:



Notice that edges 13 and 24 are traversed twice in the walk because the face is on both sides of these edges. The faces for Figure 10.26(b) has the faces:



Since one face is a cylinder which is not homeomorphic to the open disk, then this embedding is not cellular. In both embeddings of Figure 10.26, edge 13 with edges 12 and 23 cut the torus into a cylinder. In (1), edge 24 then cuts open the cylinder, but in (b) it does not.

Definition. The *Euler characteristic* of a closed surface Σ, denoted $c(\Sigma)$, is defined as:

$$c(\Sigma) = \begin{cases} 2 - 2k & \text{if } \Sigma \text{ is homeomorphic to } S_k \\ 2 - k & \text{if } \Sigma \text{ is homeomorphic to } N_k. \end{cases}$$

Note. The next result is a generalization of Euler's Formula (Theorem 10.19). For a proof, see Section 3.1. "Classification of Surfaces" in Bojan Mohar and Carsten Thomassen's *Graphs on Surfaces*, Baltimore: Johns Hopkins University Press (2001).

Theorem 10.37. Let $\tilde{G}$ be a (cellular) embedding of a connected graph G on a surface Σ. Then $v(\tilde{G}) - e(\tilde{G}) + f(\tilde{G}) = c(\Sigma)$.

Note. The sphere S_0 has Euler characteristic $2 - 2k = 2 - 2(0) = 2$. By Theorem 10.4, a graph is embeddable on the plane if and only if is embeddable on the sphere. Also, the stereographic projection used in the proof of Theorem 10.4 maps faces on the sphere bijectively to faces on the plane, so Theorem 10.37 implies for both the plane and the sphere that $v(G) - e(G) + f(G) = 2$ and in this way Theorem 10.37 generalizes Euler's Formula.

Note. The following two corollaries follow easily from Theorem 10.37 and the proof is to be given in Exercise 10.6.3.

Corollary 10.38. All (cellular) embeddings of a connected graph on a given surface have the same number of faces.

Corollary 10.39. Let G be a simple connected graph that is (cellularly) embeddable on a surface Σ. Then $m \leq 3(n - c(\Sigma))$.

Note. In Corollaries 10.23 and 10.24, K_5 and $K_{3,3}$ are shown to be nonplanar using Euler's Formula. Similarly, Theorem 10.37 (or its corollaries) can be used to show that certain graphs are not cellularly embeddable on a surface (see Exercise 10.6.4).

Note. Just as a dual of a planar graph can be defined for a given planar embedding, we can also define the dual of a graph given a cellular embedding on a surface. Figure 10.27 illustrates the dual relationship of K_6 (where the vertices are represented by open circles) and the Petersen graph (where the vertices are represented by closed disks) on the projective plane (remember the interpretation of the projective plane diagram as discussed in connection with Figure 10.25).

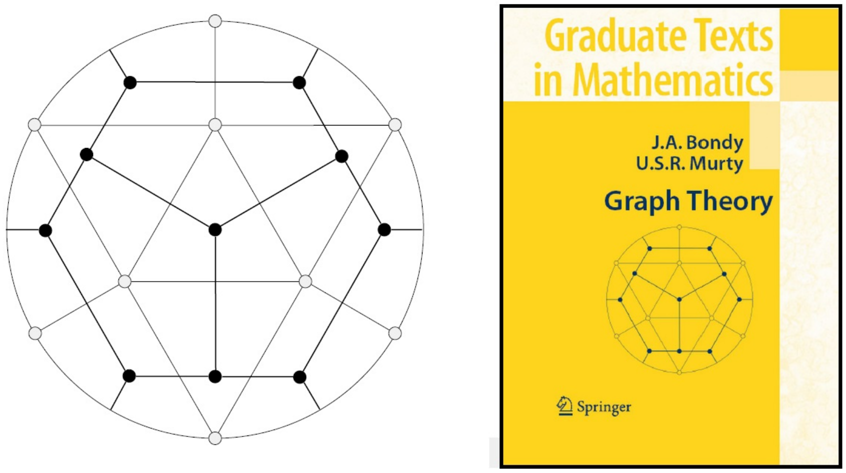


Figure 10.27 (and book cover). Dual embeddings of K_6 (with vertices represented by open circles) and the Petersen graph (with vertices represented by closed disks).

Compare this to the middle drawing of the Petersen graph in Figure 1.9 in order to recognize the Petersen graph. Notice that this is the image on the cover of the text book.

Definition. An embedding $\tilde{G}$ of a graph G on a surface is a *circular embedding* if all the faces of $\tilde{G}$ are bounded by cycles.

Note. We argued in Section 9.2 that all faces of a loopless 2-connected plane graph are bounded by cycles. This does not hold for other surfaces, as we see in the embedding of K_4 on the torus given in Figure 10.26(b) (the noncellular embedding). Related to this is the following conjecture due to F. Jaeger (it appeared in "A Survey of the Cycle Double Cover Conjecture," in *Cycles in Graphs (Burnaby, B.C., 1982)*, 1–12, North-Holland Mathematical Studies, Volume 115, Amsterdam: North Holland.

Conjecture 10.40. The Orientable Embedding Conjecture.

Every loopless 2-connected graph has a circular embedding on some orientable surface.

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