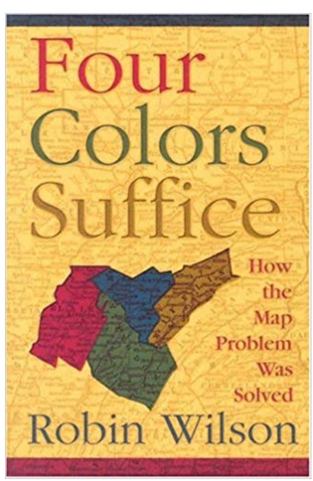
Chapter 11. The Four-Colour Problem

Note. The two most historically important results in graph theory are probably the Königsberg Bridge Problem (described in Section 3.3 in the context of Euler tours) and the four-colour problem. The Four-Colour Problem simply involves showing that for any map of countries, the countries can be assigned colours such that any two countries sharing a border have difference colours.

Section 11.1. Colourings of Planar Maps

Note. In this section we give a bit of history of the Four-Colour Conjecture/Theorem. We *discuss* the proof of the theorem, but only give a proof of Tait's Theorem which relates the 4-colourability of certain plane graphs to the edge colourability of the graph.

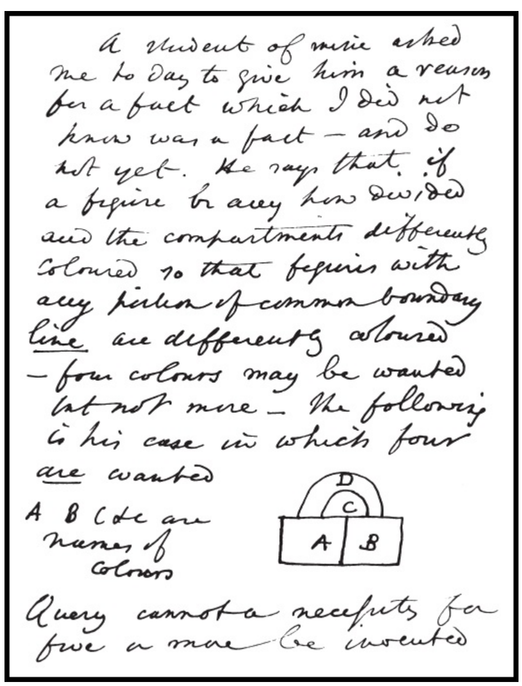
Note. For much of the history of the problem, we reference Robin Wilson's *Four Colors Suffice: How the Map Problem was Solved*, Princeton: Princeton University Press (2002); we refer to this reference simply as "Wilson." We also briefly mentioned this source in 1.7. Further Reading.



A very detailed history based on Wilson's book is given in Supplement. The Four-Colour Theorem: A History, Part 1 and Supplement. The Four-Colour Theorem: A History, Part 2. These are supplements to Section 15.2. The Four-Colour Theorem.

Note. Augustus De Morgan (1806–1871) of University College, London, and Irish mathematician William Rowan Hamilton (1805–1865) of Dublin regularly corresponded about science and mathematics. On October 23, 1852 De Morgan wrote to Hamilton about a question from one of his students:

> "A student of mine [Frederick Guthrie, brother of Francis] asked me today to give him a reason for a fact which I did not know was a fact—and do not yet. He says that if a figure be anyhow divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured—four colours may be wanted, but not more—the following is the case in which four colours are wanted. Query cannot a necessity for five or more be invented..."


A student of mine asked
me to day to give him a reason
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line are differently coloured
– four colours may be wanted
but not more – the following
is his case in which four
are wanted
A B C &c are
names of
colours

Query cannot a necessity for
five or more be invented

This close-up image of the letter is from Wilson; Chapter 2 of Wilson is available on the Princeton University Press webpage; a copy of the complete letter is available on Wikipedia's Four Color Theorem (accessed 1/16/2021). The little image with regions "coloured" A, B, C, and D show that at least four colors are necessary.

Note. According to Wilson, the first known statement of the four-colour problem in print appeared in an unsigned review (dated April 14, 1860) of William Whewell's book *The Philosophy of Discovery, Chapters Historical and Critical* in the popular literary journal *Athenaeum.* Whewell was a philosopher at Trinity College, Cambridge. In the review, the four-colour problem is described and it is claimed that it is familiar to cartographers. In 1860 De Morgan in the *Athenaeum* pointed out that this "theorem" had never been proved. This brought the problem to the attention of the American mathematician and logician Charles Sanders Peirce (1839–1914) who brought it to the attention of the American mathematical community at Harvard University. Arthur Cayley was interested in the problem, asked about it at an 1878 meeting of the London Mathematical Society and published a short note on it in the April 1879 issue of the *Proceedings of the Royal Geographic Society* in which he mentioned that he had made little progress on a proof. One contribution of Cayley was that he showed it is sufficient to consider cubic maps (in which there are exactly three countries at each meeting point; see Note 11.1.A below).

Note. London lawyer and amateur mathematician (and a student of Cayley's at Trinity College) Alfred Bray Kempe (1849–1922) presented what Wilson (see Chapter 5) describes as "the most famous fallacious proof in the whole of mathematics."

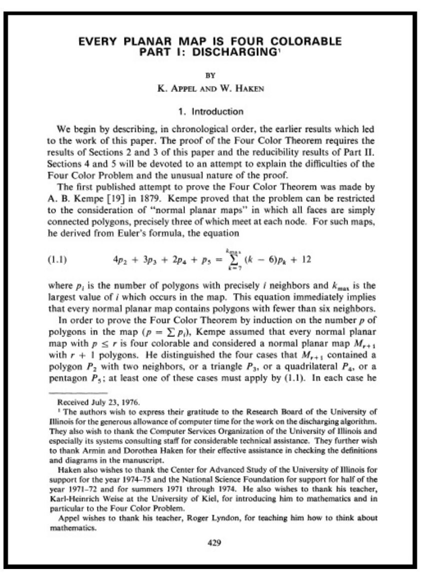
Kempe published "On the Geographical Problem of the Four Colours," *American Journal of Mathematics* **2**, 193–200 (1879) (reprinted in N. L. Biggs, E. K. Lloyd, and R. J. Wilson's *Graph Theory: 1736-1936* 2nd edition, NY: Clarendon Press, 1986). A preview of the paper appeared in *Nature* **20**, 79 (July 17, 1879) and a simplified version appeared in *Proceedings of the London Mathematical Society*, **10**, 229-31 (1878-79) and also in *Nature* **21**, 399–400 (February 2, 1880). In this, Kempe presents his "proof" of the four-colour problem. However, he makes a subtle error which goes undetected for 11 years (see Chapter 5 of Wilson for a fairly detailed but nontechnical discussion). Peter Guthrie Tait (1831–1901), influenced by Cayley and Kempe, took an interest in the problem and in 1880 found a relationship between face colourings and edge colourings (stated below as Theorem 11.4). Tait was of the opinion at the time that, following an easy inductive proof of a certain lemma, his result would imply the Four-Colour Theorem. Ultimately, his lemma ended up being as complicated to prove as the Four Colour Theorem itself (see Chapter 6 of Wilson); it is stated as Conjecture 11.5 below.

Note. Detection of the error in Kempe's paper is credited to Percy Heawood (1861–1955), a mathematics lecturer at Durham College. Heawood's paper "Map-Colour Theorem," *Quarterly Journal of Mathematics*, **24**, 332–38 (1890) (partially reprinted in *Graph Theory: 1736-1936*) states: "The present article does not profess to give a proof of [The Four-Colour Theorem]; in fact its aims are so far rather destructive then constructive, for it will be shown that there is a defect in the now apparently recognized proof [of Kempe]." Kempe himself admitted the error in *Proceedings of the London Mathematical Society*, **22**, 263 (1890-91) on April 9, 1891 at a meeting of the society (see Chapter 7 of Wilson). Heawood goes on to give a proof that five colours are sufficient to colour a map (in fact, he employed Kempe's approach in his proof). We'll explore the Five-Colour Theorem in the next section.

Note. Kenneth Appel (1932–2013) and Wolfgang Haken (1928–2022) of the University of Illinois in June 1976 announced a successful proof of the Four Color Theorem. They, along with John Koch of Wilkes University (he still has an active faculty webpage, accessed 1/16/2021) who assisted with computational work, presented the results in two papers:

1. K. Appel and W. Haken, "Every Planar Map is Four Colorable, Part I: Discharging," *Illinois Journal of Mathematics* **21**, 429–90 (1977); available on Project Euclid (accessed 1/16/2021), and

2. K. Appel, W. Haken, and J. Koch "Every Planar Map is Four Colorable, Part II: Reducibility," *Illinois Journal of Mathematics* **21**, 491–567 (1977); available on Project Euclid (accessed 1/16/2021).



EVERY PLANAR MAP IS FOUR COLORABLE
PART I: DISCHARGING[1]

BY

K. APPEL AND W. HAKEN

1. Introduction

We begin by describing, in chronological order, the earlier results which led to the work of this paper. The proof of the Four Color Theorem requires the results of Sections 2 and 3 of this paper and the reducibility results of Part II. Sections 4 and 5 will be devoted to an attempt to explain the difficulties of the Four Color Problem and the unusual nature of the proof.

The first published attempt to prove the Four Color Theorem was made by A. B. Kempe [19] in 1879. Kempe proved that the problem can be restricted to the consideration of "normal planar maps" in which all faces are simply connected polygons, precisely three of which meet at each node. For such maps, he derived from Euler's formula, the equation

$$4p_2 + 3p_3 + 2p_4 + p_5 = \sum_{k=7}^{k_{\max}} (k-6)p_k + 12 \tag{1.1}$$

where p_i is the number of polygons with precisely i neighbors and $k_{\max}$ is the largest value of i which occurs in the map. This equation immediately implies that every normal planar map contains polygons with fewer than six neighbors.

In order to prove the Four Color Theorem by induction on the number p of polygons in the map ($p = \sum p_i$), Kempe assumed that every normal planar map with $p \leq r$ is four colorable and considered a normal planar map M_{r+1} with $r+1$ polygons. He distinguished the four cases that M_{r+1} contained a polygon P_2 with two neighbors, or a triangle P_3, or a quadrilateral P_4, or a pentagon P_5; at least one of these cases must apply by (1.1). In each case he

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[1] The authors wish to express their gratitude to the Research Board of the University of Illinois for the generous allowance of computer time for the work on the discharging algorithm. They also wish to thank the Computer Services Organization of the University of Illinois and especially its systems consulting staff for considerable technical assistance. They further wish to thank Armin and Dorothea Haken for their effective assistance in checking the definitions and diagrams in the manuscript.

Haken also wishes to thank the Center for Advanced Study of the University of Illinois for support for the year 1974–75 and the National Science Foundation for support for half of the year 1971–72 and for summers 1971 through 1974. He also wishes to thank his teacher, Karl-Heinrich Weise at the University of Kiel, for introducing him to mathematics and in particular to the Four Color Problem.

Appel wishes to thank his teacher, Roger Lyndon, for teaching him how to think about mathematics.

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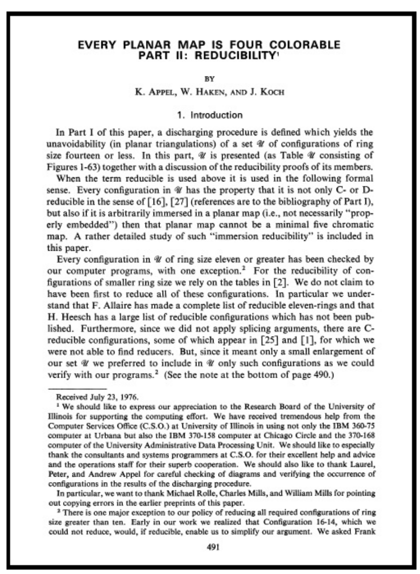

EVERY PLANAR MAP IS FOUR COLORABLE
PART II: REDUCIBILITY[1]

BY

K. APPEL, W. HAKEN, AND J. KOCH

1. Introduction

In Part I of this paper, a discharging procedure is defined which yields the unavoidability (in planar triangulations) of a set $\mathcal{U}$ of configurations of ring size fourteen or less. In this part, $\mathcal{U}$ is presented (as Table $\mathcal{U}$ consisting of Figures 1-63) together with a discussion of the reducibility proofs of its members.

When the term reducible is used above it is used in the following formal sense. Every configuration in $\mathcal{U}$ has the property that it is not only C- or D-reducible in the sense of [16], [27] (references are to the bibliography of Part I), but also if it is arbitrarily immersed in a planar map (i.e., not necessarily "properly embedded") then that planar map cannot be a minimal five chromatic map. A rather detailed study of such "immersion reducibility" is included in this paper.

Every configuration in $\mathcal{U}$ of ring size eleven or greater has been checked by our computer programs, with one exception.[2] For the reducibility of configurations of smaller ring size we rely on the tables in [2]. We do not claim to have been first to reduce all of these configurations. In particular we understand that F. Allaire has made a complete list of reducible eleven-rings and that H. Heesch has a large list of reducible configurations which has not been published. Furthermore, since we did not apply splicing arguments, there are C-reducible configurations, some of which appear in [25] and [1], for which we were not able to find reducers. But, since it meant only a small enlargement of our set $\mathcal{U}$ we preferred to include in $\mathcal{U}$ only such configurations as we could verify with our programs.[2] (See the note at the bottom of page 490.)

Received July 23, 1976.

[1] We should like to express our appreciation to the Research Board of the University of Illinois for supporting the computing effort. We have received tremendous help from the Computer Services Office (C.S.O.) at University of Illinois in using not only the IBM 360-75 computer at Urbana but also the IBM 370-158 computer at Chicago Circle and the 370-168 computer of the University Administrative Data Processing Unit. We should like to especially thank the consultants and systems programmers at C.S.O. for their excellent help and advice and the operations staff for their superb cooperation. We should also like to thank Laurel, Peter, and Andrew Appel for careful checking of diagrams and verifying the occurrence of configurations in the results of the discharging procedure.

In particular, we want to thank Michael Rolle, Charles Mills, and William Mills for pointing out copying errors in the earlier preprints of this paper.

[2] There is one major exception to our policy of reducing all required configurations of ring size greater than ten. Early in our work we realized that Configuration 16-14, which we could not reduce, would, if reducible, enable us to simplify our argument. We asked Frank

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Their approach was to argue that if the Four-Colour Conjecture is false, then there is a smallest planar graph (in terms of the number of faces) that requires five colours. In looking for such a counterexample, they considered two ideas. First, that such a counterexample has a *unavoidable set*, similar to the role that K_5 and $K_{3,3}$ play in the setting of planar graphs. Second, a *reducible configuration* is an arrangement of faces of a planar graph that cannot occur in a minimal counterexample; if a graph contains such a configuration then the graph can be reduced to a smaller graph (hence the name of the second paper) which is checked for four colourability. If the smaller graph is four colourable then so is the original graph, and if it is not four colourable then the original graph is not minimal. The process of checking for four colourability involves "discharging rules" (hence the name of the first paper). The original papers involved 487 discharging rules and 1482 reducible configurations (and the papers were supplemented with a microfiche containing 450 pages of further diagrams and details). See Chapters 9 and 10 of Wilson. Some of the history is from the Wikipedia webpage on The Four Color Theorem (accessed 1/16/2021).

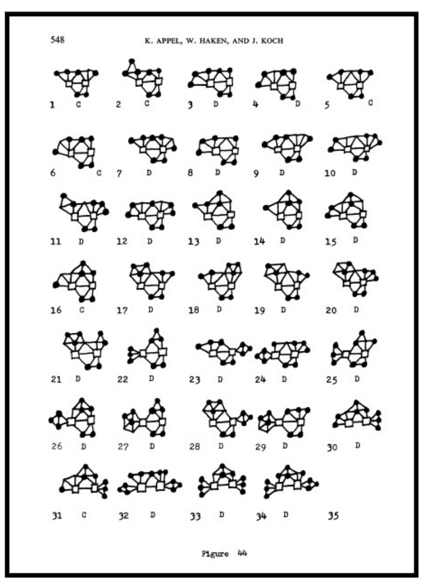

Figure 44 of Part II. Some reducible configurations.

Note. The heavy reliance on computers in Appel and Haken's proof was immediately a topic of discussion and concern in the mathematical community. The issue was the fact that no individual could check the proof; of special concern was the reductibility part of the proof because the details were "hidden" inside the computer. Though it isn't so much the *validity* of the result, but the *understanding* of the proof. Appel himself commented: "...there were people who said, 'This is terrible mathematics, because mathematics should be clean and elegant,' and I would agree. It would be nicer to have clean and elegant proofs." See page 222 of Wilson. By the early 1980s rumors of errors in Appel and Haken's work were circulating. In 1981 a master's student, Ulrich Schidt, at Aachen Technische Hochschule found an error that took Haken two weeks to correct. A few misprints were also found. A minor error in one configuration was found in 1985. Other than this and few additional misprints, no significant error has been found. Appel and Haken published *Every Planar Map is Four Colorable*, Contemporary Mathematics Volume 98, American Mathematical Society (1989), a 741 page work that gave more details of the proof, proved some related results, corrected all the known errors, and included a printed version of the microfiche pages.

Note. In 1994, Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas gave their own proof using the same general approach that Appel and Haken used. However, their work involved only 633 reducible configurations (versus Appel and Haken's 1482) and only 32 discharging rules (versus Appel and Haken's 487). Their work was described in "The Four-Colour Theorem," *Journal of Combinatorial Theory, Series B*, **70**, 2–44 (1997). We now return to Bondy and Murty and introduce the definitions we need to translate the Four-Colour Theorem into specific graph theoretic terms.

Definition. A *k-face colouring* of a plane graph is an assignment of k colours to its faces. The colouring is *proper* if no two adjacent faces are assigned the same colour. A plane graph is *k-face colourable* if it has a proper k-face colouring.

Note. Since every "map" can be thought of as a plane graph without cut edges (a cut edge would represent a boarder between a country and itself so that a plane graph with cut edges cannot have a proper face colouring), then the Four-Colour Conjecture is equivalent to the following.

Conjecture 11.1. The Four-Colour Conjecture (Face Version).
Every plane graph without cut edges is 4-face colourable.

Note. We can now state Appel and Haken's result ("The Four-Colour Theorem") as follows.

Theorem 11.2. The Four-Colour Theorem.
Every plane graph without cut edges is 4-face colourable.

Note. We now look at vertex colourings and edge colourings, and translate The Four-Colour Conjecture into these settings.

Definition. A *k-vertex-colouring* of a graph, or simply a *k-colouring*, is an assignment of k colours to its vertices. The colouring is *proper* if no two adjacent vertices are assigned the same colour. A graph is *k-colourable* if it has a proper k-colouring.

Note. Since adjacent pairs of vertices of a plane graph correspond to adjacent faces of its dual, the Four-Colour Problem is equivalent to the problem of finding a 4-colouring of every loopless plane graph. Notice that this does not refer to any particular embedding. This gives the following version of The Four-Colour Conjecture.

Conjecture 11.3. THE FOUR-COLOUR CONJECTURE (VERTEX VERSION).
Every loopless planar graph is 4-colourable.

Note 11.1.A. To establish the vertex version of The Four-Colour Conjecture, it suffices to show that all *simple* connected planar graphs are 4-colourable (since any parallel arcs can be subdivided and the new vertices 4-coloured as needed). In Exercise 11.1.1/11.2.1 (different printings of the text book have slight differences, in particular in the location of exercises in this chapter) it is to be shown that the Four-Colour Conjecture is true provided that it is true for all simple 3-connected maximal planar graphs. By Corollary 10.21, a simple planar graph on at least three vertices satisfies $m \leq 3n - 6$, so a maximal (in terms of edges) simple planar graph satisfies $m = 3n - 6$ which implies (also be Corollary 10.21) that every planar embedding is a triangulation. So the Four-Colour Conjecture is equivalent to the claim that every 3-connected triangulation is 4-colourable. Note the dual of a triangulation is cubic by Proposition 10.11 and by Exercise 10.2.9 the dual of a simple 3-connected plane graph is also simple and 3-connected. So by considering the dual, the Four-Colour Conjecture is also equivalent to the claim that every 3-connected cubic plane graph is 4-face-colourable.

Definition. A *k-edge-colouring* of a graph is an assignment of k colours to its edges The colouring is *proper* if no two adjacent edges are assigned the same colour. A graph is *k-edge-colourable* if it has a proper k-edge colouring.

Theorem 11.4. Tait's Theorem.
A 3-connected cubic plane graph is 4-face-colourable if and only if it is 3-edge-colourable.

Note. By Note 11.1.A and Tait's Theorem, we have another version of the Four-Colour Conjecture.

Conjecture 11.5. The Four-Colour Conjecture (Edge Version).
Every 3-connected cubic planar graph is 3-edge-colorable.

Note. If a cubic graph (in which every vertex is of degree 3) is Hamiltonian (in which case n is even by Theorem 1.1) then the edges of the Hamilton cycle can be properly coloured with two colours (by just alternating the edge colours on the cycle). The remaining edges can be coloured with a third colour so that such a graph is 3-edge-colourable. So if every 3-connected cubic planar graph is Hamiltonian, then Conjecture 11.5 would hold. Peter Tait thought this result would follow easily (as commented above) so that his Theorem 11.4 would imply the Four-Colour Conjecture. However, it was shown by Tutte in 1946 that a non-Hamiltonian 3-connected cubic planar graph exists; see W. T. Tutte, "On Hamilton Cycles," *Journal of the London Mathematical Society*, **21**, 98–101 (1946).

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