Section 11.2. The Five-Colour Theorem

Note. As discussed in the previous section, Alfred Kempe published an incorrect proof of The Four-Colour Theorem in 1879. Percy Heawood pointed out the error in 1890. However, Heawood was able to use Kempe's ideas to prove that any map can be properly five coloured. Heawood's proof is inductive on the number of vertices of the planar graph and is basically the proof we present now.

Theorem 11.6. The Five-Colour Theorem.
Every loopless planar graph is 5-colourable.

Note. Another proof of the Five-Colour Theorem is outlined in Exercise 11.2.1/11.2.6. A proof using list colourings is given in Chapter 15, "Colourings of Maps" (see Corollary 15.9).

Note. Some of the ideas used in the proof of The Four-Colour Theorem are given in Section 15.2, "The Four-Colour Theorem." In particular, reducibility, avoidable configurations, and discharging rules are discussed.

Note. The Four-Colour Theorem addresses maps on the plane (or, as approached here, planar graphs). As in Section 10.6, "Surface Embeddings of Graphs," we can ask an analogous question about maps on surfaces (or graphs embedded on surfaces). This is considered in Section 15.1, "Chromatic Numbers of Surfaces." The *chromatic number* of a surface Σ is defined as the least integer k such that every graph embeddable on surface Σ is k-colourable. So for the plane, the chromatic number is 4. A bound on the chromatic number of a surface is given in terms of the Euler characteristic of the surface in Heawood's Inequality (see Theorem 15.1).

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