Chapter 12. Stable Sets and Cliques

Note. We considered stable sets and cliques briefly in Chapter 8 in the context of of computational complexity. In Section 12.1 we define stable sets, define some parameters related to stable sets, discuss bounds on these parameters, and give an application. In Section 12.2 we define a Turán graph and use it to put a bound on the number of edges in a clique-free graph (in Turán's Theorem, Theorem 12.7). In Section 12.3 we define Ramsey graphs and Ramsey numbers, and put bounds on Ramsey numbers. In Section 12.4 we define a regular partition of the vertex set of a graph and give a condition under which such a partition exists (in the Regularity Lemma, Theorem 12.16). We go through the lengthy proof of the Regularity Lemma and, in the process, introduce linear Ramsey numbers. All graphs in this chapter are assumed to be simple.

Section 12.1. Stable Sets

Note. In this section we define a stable set (which we originally considered in Chapter 8, and we'll see again in the next chapter; see Exercise 13.1.1). We define the stability number, the covering number, and the clique number. We consider path partitions of digraphs and directed paths orthogonal to stable sets, and relate the size of a path partition to the stability number (in the Gallai-Milgram Theorem, Theorem 12.2). We define a kernel in a digraph in terms of domination, and give conditions under which a digraph has a kernel (in Richardson's Theorem, Theorem 12.6).

Definition. A *stable set* (or *independent set*) in a graph is a set of vertices, no two of which are adjacent. A stable set in a graph is *maximum* if the graph contains no larger stable set and *maximal* if the set cannot be extended to a larger stable set. The cardinality of a maximum stable set in a graph G is the *stability number* of G, denoted $\alpha(G)$.

Note. Figure 12.1(a) shows a maximal stable set on three vertices in the Petersen graph and Figure 12.1(b) shows a maximum stable set on four vertices in the Petersen graph (for the Petersen graph P, $\alpha(P) = 4$).

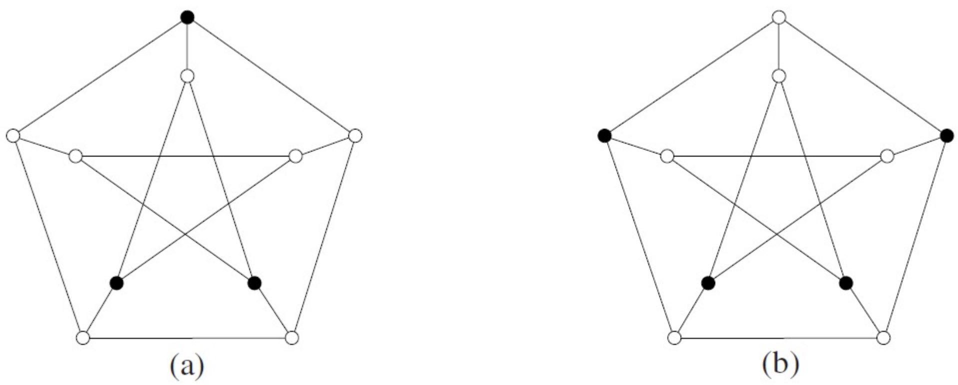


Fig. 12.1. (a) A maximal stable set, and (b) a maximum stable set

Definition. An *edge covering* of a graph is a set of edges which together meet all vertices of the graph. A *covering* of a graph is a set of vertices which together meet all edges of the graph. The minimum number of vertices in a covering of a graph G is the *covering number* of G, denoted $\beta(G)$.

Note. The vertices represented by open circles in Figure 12.1 are examples of coverings of the Petersen graph. Notice that the use of the term "covering" here is unrelated to the use in Section 2.4. Decompositions and Coverings (where both decompositions and coverings are sets of subgraphs of a given graph).

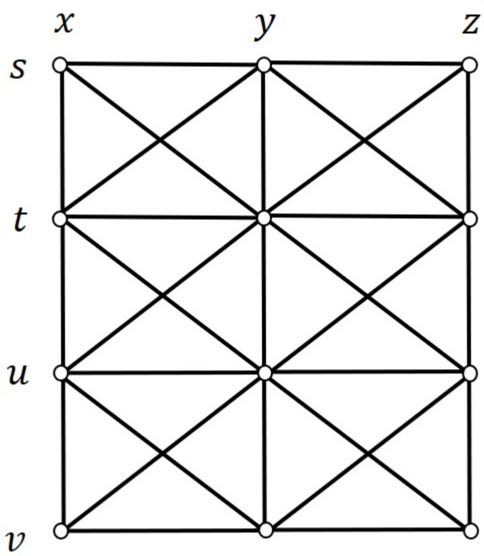
Note. In Exercise 12.1.2, it is to be shown that S is a stable set of a graph G is and only if $V \setminus S$ is a covering of G. Therefore the stability number of a graph plus the covering number of the graph must equal the total number of vertices of the graph: $\alpha(G) + \beta(G) = v(G)$.

Definition. A *clique* of a graph G is a set of mutually adjacent vertices (i.e., the vertices of a complete subgraph of G). The maximum size of a clique of a graph G is the *clique number* of G, denoted $\omega(G)$.

Note. Notice that a set of vertices S is a clique of a simple graph G is and only if it is a stable set of the complement graph $\overline{G}$. See Figure 12.1 again for insight in understanding this claim. Therefore, we have that the clique number of G equals the stability number of $\overline{G}$: $\omega(G) = \alpha(\overline{G})$. So any assertaion about stable sets can be restated in terms of cliques (or coverings). In Chapter 8, we considered the computation complexities of finding a maximum stable set; see my online notes (in preparation) for Mathematical Modeling Using Graph Theory (MATH 5870).

Definition. For simple graph G and H, the *strong product*, $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$, where vertices (u, x) and (v, y) are adjacent if and only if (1) $uv \in E(G)$ and $x = y$, (2) $u = v$ and $xy \in E(H)$, or (3) $uv \in E(G)$ and $xy \in E(H)$.

Examples. Let G be a 3-path and H a 2-path with, say, $V(G) = \{s, t, u, v\}$, $E(G) = \{st, tu, uv\}$, $V(H) = \{x, y, z\}$, and $E(H) = \{xy, yz\}$. The $G \boxtimes H$ is:



The vertical edges are due to (1)

The horizontal edges are due to (2)

The diagonal edges are due to (3)

Example 12.1. Transmitting Messages over a Noisy Channel.
This problem is due to Claude E. Shannon (April 30, 1916–February 24, 2001), an American mathematician and cryptographer sometimes called the "father of information theory." A transmitter over a communication channel is capable of sending signals belonging to a certain finite set (or *alphabet*) A. Some pairs of these signals are so similar to each other that they might be confounded by the receiver because of possible distortion during transmission. Given a positive integer k, what

is the greatest number of sequences of signals (or *words*) of length k that can be transmitted with no possibility of confusion at the receiving end?

We use a stable set in the strong products of graphs to model this. Let G be the graph with vertex set A (the alphabet) and with vertices u and v adjoint if (and only if) they represent signals that might be confused with each other. Let G^k be the strong product of k (the length of the words) copies of G: $G^k = \underbrace{G \boxtimes G \boxtimes \cdots \boxtimes G}_{k \text{ times}}$. Notice that the vertices of G^k are ordered k-tuples of letters from A (that is, words of length k). In G^k we have two distinct vertices (that is, words) $(u_1, u_2, \ldots, u_k)$ and $(v_1, v_2, \ldots, v_k)$ adjacent if either $u_i = v_i$ or $u_i v_i \in E(G)$ for some $1 \leq i \leq k$. That is, two distinct words are adjacent in G^k if there is the possibility that one of them might be mistaken for the other by the receiver. In a stable set of vertices of G^k, there is no concern about one word in the stable set being confused with another word of the stable set. So to answer the posed question, the greatest number of words that can be transmitted without confusion is the size of a largest stable set. That is, it is the stability number of G^k, $\alpha(G^k)$. As an example, suppose $A = \{0, 1, 2, 3, 4\}$ and each signal i may be confused with either $i - 1 \pmod 5$ or $i + 1 \pmod 5$, so that $G = C_5$. With $k = 2$, we consider $G^2 = C_5 \boxtimes C_5$, a drawing of which is given in Figure 12.2(b) with the graph drawn on a torus (so the vertices on the top row are the same as the vertices on the bottom row, and similarly for the left-most and right-most columns). A stable set is given by the five solid vertices, so $\alpha(G^2) \geq 5$ (in fact, in Exercise 12.1.8 it is to be shown that $\alpha(G^2) = 5$). So in this example, there are five words of length two that may be transmitted with no possibility of confusion under this scheme. We next turn out attention to digraphs.

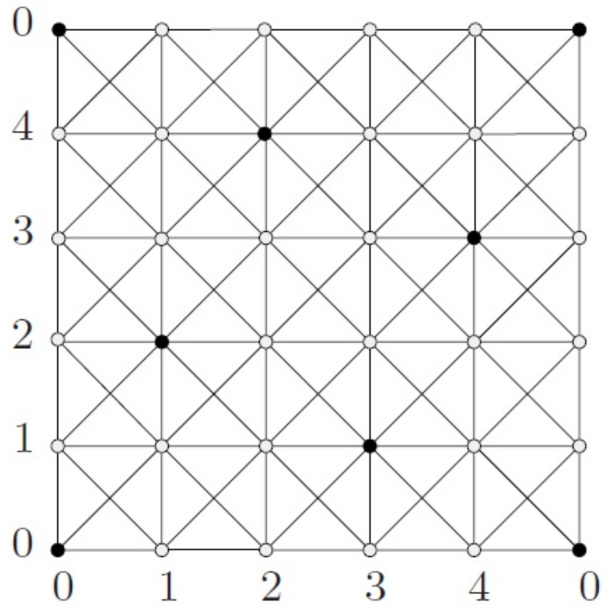


Fig. 12.2. (b) a stable set of five vertices in C_5^2

Definition. A *stable set in a digraph* is a stable set in its underlying graph. The number of vertices in a largest stable set of a digraph D is the *stability number* of D, denoted $\alpha(D)$.

Note. By Rédei's Theorem (Theorem 2.3), we know that every tournament (i.e., orientation of a complete graph) has a directed Hamilton path (i.e., a directed path containing every vertex of the digraph). We could consider for non-tournaments the question: "How many disjoint directed paths are needed to cover the vertex set of a digraph?" In 1960 Gallai and Milgram showed that the number of such directed paths is bounded by the stability number, as we now explain.

Definition. A covering of the vertex set of a graph or digraph by disjoint paths or directed paths is a *path partition*. A path partition of a graph or digraph with the least number of paths is an *optimal partition*. The number of paths in an optimal partition of a digraph D is denoted $\pi(D)$.

Theorem 12.2. The Gallai-Milgram Theorem.
For any digraph D, $\pi(D) \leq \alpha(D)$.

Definition. A directed path P and a stable set S are *orthogonal* if they have exactly one common vertex. A path partition $\mathcal{P}$ and a stable set S are *orthogonal* if each path in $\mathcal{P}$ is orthogonal to S.

Note. Gallai and Milgram actually proved something more general than Theorem 12.2 and which involves sets S orthogonal to a given optimal path partition $\mathcal{P}$.

Theorem 12.3. Let $\mathcal{P}$ be an optimal path partition of a digraph D. Then there is a stable set S in D which is orthogonal to $\mathcal{P}$.

Note. Since $\mathcal{P}$ is an optimal path partition of D in Theorem 2.3, then $\pi(D) = |\mathcal{P}|$. Since S is *some* stable set in D then $|S| \leq \alpha(D)$. Since each directed path in $\mathcal{P}$ shares exactly one vertex with S (by the definition of "orthogonal") then $|\mathcal{P}| \leq |S|$. Hence $\pi(D) = |\mathcal{P}| \leq |S| \leq \alpha(D)$, so that $\pi(D) \leq \alpha(D)$ and hence the Gallai-Milgram Theorem (Theorem 12.2) is implied by Theorem 12.3. We now turn to a proof of Theorem 12.3.

Lemma 12.4 Let $\mathcal{P}$ be a path partition of a digraph D. Suppose that no stable set of D is orthogonal to $\mathcal{P}$. Then there is a path partition $\mathcal{Q}$ of D such that $|\mathcal{Q}| = |\mathcal{P}| - 1$, $i(\mathcal{Q}) \subset i(\mathcal{P})$, and $t(\mathcal{Q}) \subset t(\mathcal{P})$ where $i(\mathcal{P})$ denotes the set of initial vertices of the paths in $\mathcal{P}$ and $t(\mathcal{P})$ is the set of terminal vertices of the paths in $\mathcal{P}$.

Note. The basic idea of the proof of Lemma 12.4 is to remove a terminal vertex x (in Figure 12.4), apply the induction hypothesis to get a path partition of the smaller digraph, and then add the vertex z back in (adding either arc (x, z) or arc (y, z), depending on whether x or y is a terminal vertex in the directed path partition given by the induction step). Since the proof is inductive, in order to use it to find a path partition $\mathcal{P}$ and a stable set S orthogonal to $\mathcal{P}$ we need to do so recursively. In Exercise 12.1.9, a polynomial time algorithm is to be given, based on the proof of Lemma 12.4, for finding in digraph D a directed path partition $\mathcal{P}$ and a stable set S, orthogonal to $\mathcal{P}$, such that $|\mathcal{P}| = |S|$ (notice that this last cardinality condition is redundant since S and $\mathcal{P}$ are orthogonal if, by definition, each directed path $P \in \mathcal{P}$ intersects S is a single vertex). This algorithm acts as **a proof of Theorem 12.3** (and recall that Theorem 12.3 then implies the Gallai-Milgram Theorem, Theorem 12.2). An application of the Gallai-Milgram Theorem to partially ordered sets (see Section 2.1. Subgraphs and Supergraphs) is the following.

Theorem 12.5 Dilworth's Theorem.

The minimum number of chains into which the elements of a partially ordered (finite) set P can be partitioned is equal to the maximum number of elements in an antichain of P.

Note. If S is a maximal stable set in a graph G, then every vertex in $G - S$ is adjacent to some vertex of S. We now extend this idea to digraphs. Recall that if (u, v) is an arc of a digraph then we say that u "dominates" v.

Definition. A *kernel* in a digraph D is a stable set S of D such that each vertex of $D - S$ dominates some vertex of S.

Note. In Figure 12.6, a kernel of the given digraph consists of the solid vertices. Not every digraph has kernels. Directed off length cycles are the simplest examples and, as the next result shows, there is something fundamental about these directed cycles.

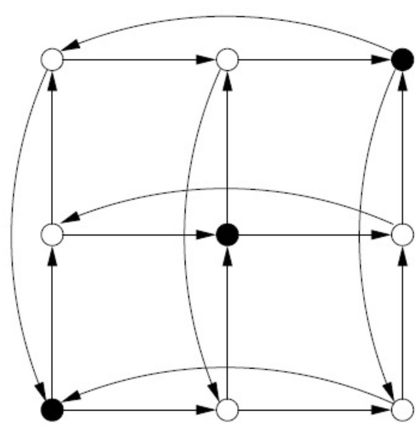


Figure 12.6

Theorem 12.6. Richardson's Theorem.

Let D be a digraph which contains no directed odd cycle. Then S has a kernel.

Note. Richardson's Theorem (Theorem 12.6) gives a condition under which a digraph has a kernel (the condition applies to acyclic digraphs, for example). For arbitrary digraphs, the decision of whether the digraph has a kernel is an NP-complete problem (as is to be shown in Exercise 12.1.16). V. Chvátal and L. Lovász introduced the idea of a semi-kernel in 1974 and proved that every digraph has a semi-kernel (see Exercise 12.1.17).

Definition. A *semi-kernel* in a digraph D is a stable set S which is reachable from every vertex of $D - S$ by a directed path of length one or two.

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