Section 12.2. Turán's Theorem

Note. In this section we consider the maximum number of edges in a graph that contains no subgraphs which are complete (other than subgraphs on 1 vertex); that is, the maximum number of edges in a clique-free graph. We use this result ("Turán's Theorem) to prove a result in combinatorial geometry concerning the distances of points between points in the Cartesian plane.

Note. We saw in Exercise 2.1.3 that a graph on n vertices contains a cycle if it has at least n edges. In Exercise 2.1.17, we saw that a graph on n vertices contains a triangle if it has more than $n^2/4$ edges (this is called "Mantel's Theorem"). Since a triangle is also a K_3, we can think of Turán's Theorem as a generalization of Mantel's Theorem in that is gives the maximum number of edges a graph on n vertices can contain without containing a clique (i.e., a copy of K_k for some $k \geq 2$).

Definition. If F is a simple graph, let $\text{ex}(n, F)$ denote the maximum number of edges in a graph G on n vertices which does not contain a copy of F. Such a graph G is an *extremal graph* (for this property of not containing a copy of F), and the set of extremal graphs is denoted $\text{Ex}(n, F)$.

Definition. The simple complete k-partite graph on n vertices in which all parts are as equal in size as possible (that is, each partite set is of size $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$) is a *Turán graph*, denoted $T_{k,n}$. (See also Exercise 1.1.11.)

Note. Turán's Theorem was first given by Paul Turán in "Eine Extremalaufgabe aus der Graphentheorie" [in Hungarian; "On an extremal problem in graph theory"] *Matematikai és Fizikai Lapok* **48**, 436–452 (1941). The proof of Turán's Theorem we give here is due to A. A. Zykov in "On Some Properties of Linear Complexes" [in Russian] *Matematicheskii Sbornik Novaya Seriya* **24**(66), 13–188 (1949).

Theorem 12.3. Turán's Theorem.
Let G be a simple graph which contains no K_k, where $k \geq 2$. Then $e(G) \leq e(T_{k-1,n})$, with equality if and only if $G \cong T_{k-1,n}$.

Note. In the proof of Turán's Theorem, we saw that if graph G has n vertices and more than $e(T_{k,n})$ edges and if v is a vertex of maximum degree Δ in G, then the subgraph $G[N(v)]$ induced by the neighbors of G has more than $t_{k-2}(\Delta)$ edges. We can iterate this procedure to find a clique of size k in G. The algorithm (a greedy algorithm; see Section 8.5. Greedy Heuristics for definitions) is given in Exercise 12.2.4 as follows:

INPUT: Let G be a graph on n vertices and more than $t_{k-1}(n)$ edges, where

$$t_{k-1}(n) = e(T_{k-1,n}).$$

OUTPUT: A clique S of k vertices.

1: set $S = \varnothing$ and $i = 1$

2: **while** $i < k$ **do**

3: select a vertex v_i of maximum degree in G

4: replace S by $S \cup \{v_i\}$, G by $G[N(v_i)]$ and i by $i + 1$

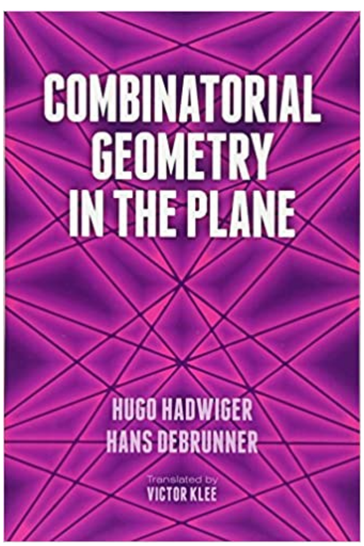
5: **end while**

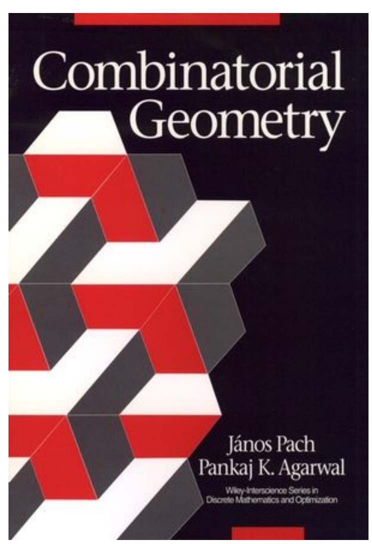
6: select a vertex v_k of maximum degree in G

7: replace S by $S \cup \{v_k\}$

8: return S

Note. We next consider an application of Turán's Theorem to combinatorial geometry. Combinatorial geometry deals with combinations and arrangements of geometric objects. Common topics are packings, tilings, symmetry, and decompositions. Perhaps the most famous problem in combinatorial geometry is Kepler's Conjecture on the optimum packing of spheres. Two classic books in combinatorial geometry are Hugo Hardwiger and Hans Debrunner's (translated by Victor Klee) *Combinatorial Geometry in the Plane*, Holt Rineheart & Winston (1964), and János Pach and Rankaj Agarwal's *Combinatorial Geometry*, Wiley-Interscience SEries in Discrete Mathematics and Optimization (1995). Hardwiger and Debrunner's book is still in print by Dover Publications; Pach and Agarwal's original book is still in print.





Definition. The *diameter* of a set of points in the plane is the maximum distance (measured in the usual Euclidean distance) between two points of the set.

Note. We will consider sets of diameter one. If there are lots of points in such a set then we would expect some points to be close together; recall the Weierstrass-Bolzano Theorem states that every bounded infinite set of real numbers has at least one limit point (so that there are points in the set that are arbitrarily close together). See my online notes for Analysis 1 (MATH 4217/5217) on Section 2.3. Bolzano-Weierstrass Theorem. We consider $d \in (0, 1)$ and ask how many points in $\{x_1, x_2, \ldots x_n\}$ can be at distance greater than d. Here, we consider $d = 1/\sqrt{2}$ and give a result of Paul Erdös from the mid 1950s.

Note 12.2.A. Consider the case of $n = 6$ points, and the two configurations given in Figure 12.8. In Figure 12.8(a), the points are placed on a regular pentagon of diameter 1. The three pairs of points, (x_1, x_4), (x_2, x_5), and (x_3, x_6) are at a distance 1 apart. The pairs of points (x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5), (x_5, x_6), and (x_6, x_1). are distance $1/2 < 1/\sqrt{2}$ apart. The pairs of points (x_1, x_3), (x_2, x_4), (x_3, x_5), (x_4, x_6), (x_5, x_1), and (x_6, x_2) are a distance $\sqrt{3} > 1/\sqrt{2}$ apart. This covers all $\binom{6}{2} = 15$ pairs and 9 of them are distance greater than $1/\sqrt{2}$ apart. However, in Figure 12.8(b) 12 of the pairs of points are a distance greater than $1/\sqrt{2}$ (all but (x_1, x_2), (x_3, x_4), and (x_5, x_6)). In fact, this is the best possible for $n = 6$ points. We now give a general result.

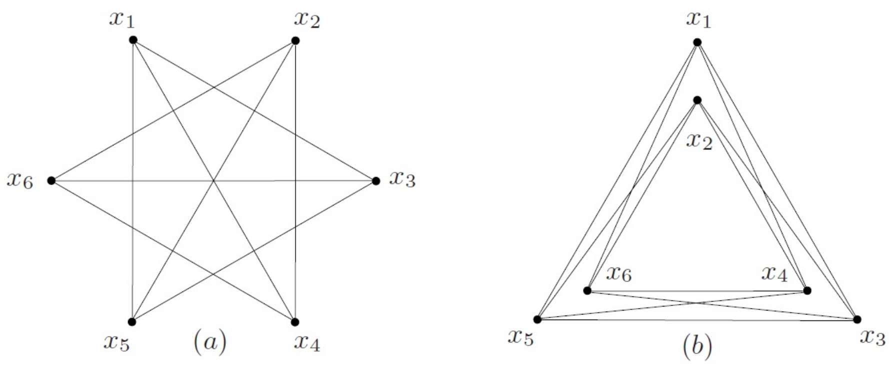


Figure 12.8

Theorem 12.4. Let S be a finite set of diameter one in the plane. Then the number of pairs of points of S whose distance is greater than $1/\sqrt{2}$ is at most $\lfloor n^2/3 \rfloor$, where $n = |S|$. Moreover, for each $n \geq 2$, there is a set of n points of diameter one in which exactly $\lfloor n^2/3 \rfloor$ pairs of points are at distance greater than $1/\sqrt{2}$.

Note. In the example considered in Note 12.2.A, we had $n = 6$. By Theorem 12.4, with $n = 6$ we have at most $\lfloor n^2/3 \rfloor = \lfloor 6^2/3 \rfloor = 12$ pairs of points at distance greater than $1/\sqrt{2}$, so the 12 pairs of points given in the example are now known to be the most possible.

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