Section 12.3. Ramsey's Theorem

Note. A clique (i.e., a complete subgraph) of a graph G corresponds to a stable set (i.e., an independent set) in the complement graph $\overline{G}$. So that we expect graphs with large cliques to have small stable sets and conversely. The main result of this section is due to Frank P. Ramsey and appeared in "On a Problem of Formal Logic," *Proceedings of the London Mathematical Society*, **30**(1), 264–286 (1930). He proves that, given any positive integers k and ℓ, there exists a smallest integer $r(k, \ell)$ such that every graph on $r(k, \ell)$ vertices contains either a clique on k vertices or a stable set on ℓ vertices. It is to be shown in Exercise 12.3.1 that $r(k, \ell) = r(\ell, k)$ (by comparing G and $\overline{G}$).

Definition. The smallest integer $r(k, \ell)$ such that every graph on $r(k, \ell)$ vertices contains either a clique on k vertices or a stable set of ℓ vertices is a *Ramsey number*. When $k = \ell$ these are *diagonal Ramsey numbers*.

Note 12.A. We have $r(1, \ell) = r(k, 1) = 1$ since K_1 contains a clique on 1 vertex. Also, $r(2, \ell) = \ell$ because any graph on ℓ vertices contains either a stable set on $k = 2$ vertices (that is, there are 2 vertices not adjacent) or is a complete graph (and so has a clique of size ℓ).

Note 12.B. These same ideas are addressed in Introduction to Graph Theory (MATH 4347/5347) in Section 4.3. Ramsey Theory. But there these ideas are addressed in terms of two colorings of the edges of complete graphs. Then a clique can be thought of as a red complete subgraph and an independent set can be though of as a blue complete subgraph. The "Theorem on Friends and Strangers" is presented in these notes and relates to a party where there are strangers and acquaintances. In fact, the next result appears in this source as Theorem 4.3.2. A proof is given in Introduction to Graph Theory that is inductive on $k+\ell$. The proof given here is similar (a vertex v is pulled aside and a smaller graph considered), though it is not written explicitly as an inductive proof.

Theorem 12.9. For any two integers $k \geq 2$ and $\ell \geq 2$,

$$r(k,\ell) \leq r(k,\ell-1) + r(k-1,\ell).$$

Furthermore, if $r(k,\ell-1)$ and $f(k-1,\ell)$ are both even, strict inequality holds in the inequality.

Note. Frank P. Ramsey (February 22, 1903–January 19, 1930) was not a graph theorist. Ramsey worked in philosophy, mathematics, and economics. He was a friend of Ludwig Wittgenstein while at the Trinity College of Cambridge University (London). In his paper "On a Problem of Formal Logic," *Proceedings of the London Mathematical Society* **30**(1), 264–286 (1930) (a version is available online on William Gasarch's University of Maryland webpage; the 1928 date on the paper reflects the fact that it was read to the society on December 13, 1928), Ramsey presents the main result of this section. However, as the title of the paper suggests, this was not the main focus of the paper, though this is probably his most famous result, and the theorem appears more in passing than as the focus of his work which was addressing a decidability problem in first order logic. His related work on a theory of types (related to the work of Russell and Whitehead in their three volume *Principia Mathematica* of the early 1900s; this work appeared as F. P. Ramsey, F.P. "The Foundations of Mathematics," *Proceedings of the London Mathematical Society*, **25**(1), 338-384 (1926)) was later used by Kurt Gödel in his work on incompleteness. For details this work in mathematical foundations, see my online notes for Introduction to Modern Geometry (MATH 4157/5157) on Section 1.6. Completeness and Categoricalness and my online presentation for Great Ideas in Science 1 and 2 (BIOL 3018 and BIOL 3028) on Introduction to Math Philosophy and Meaning. Ramsey died of complications following abdominal surgery for liver problems in 1930. He was a month shy of his 27th birthday. This biographical information is from the Wikipedia page on Frank Ramsey (accessed 5/7/2022).

Image from the MacTutor History of Mathematics Archive biography of Ramsey (accessed 5/7/2022)

Note. We will present bounds on Ramsey numbers, but few Ramsey numbers are known precisely. We can get lower bounds by considering specific graphs. Figure 12.11 gives four such graphs.

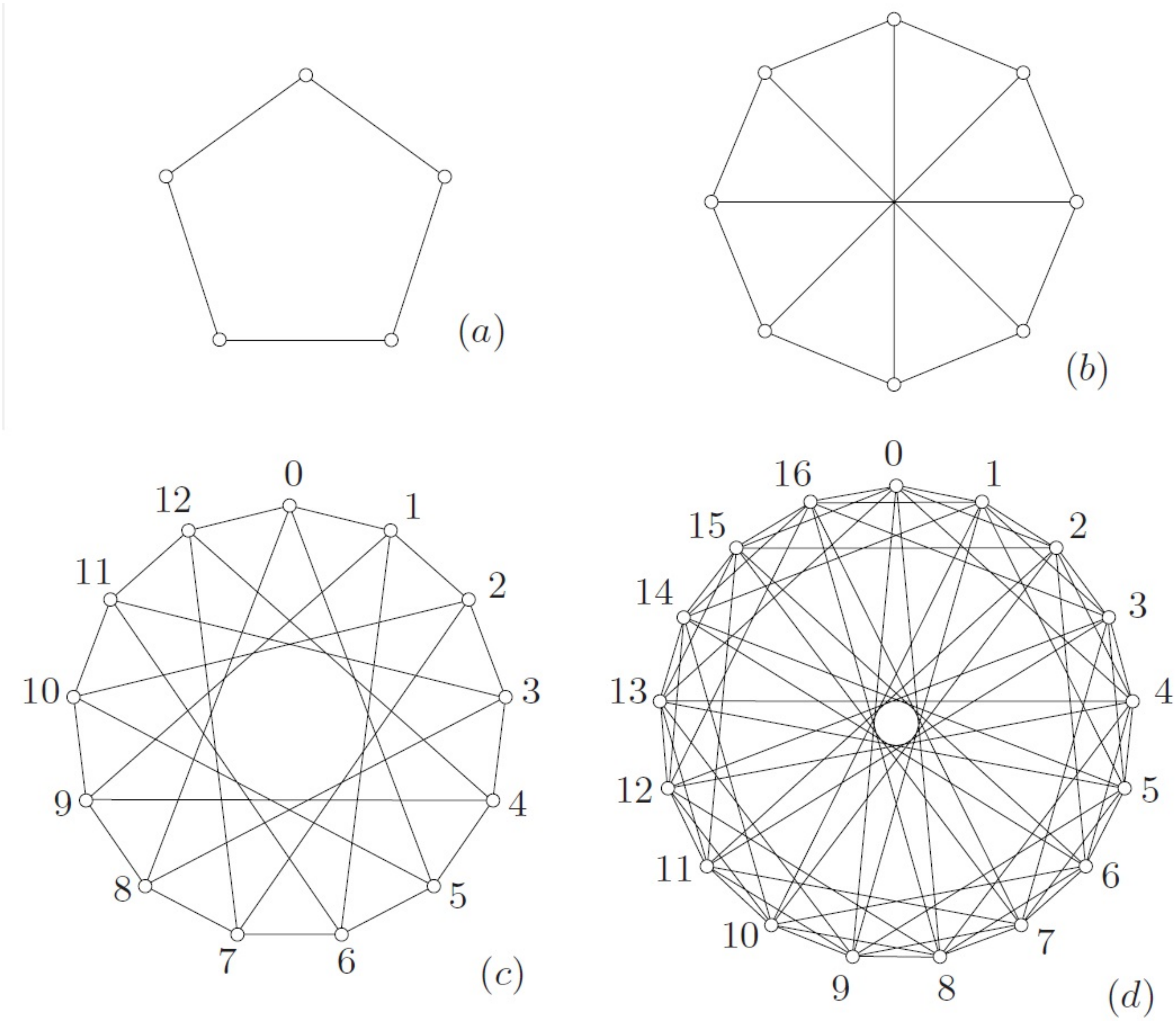

Figure 12.11

In Figure 12.11(a), we have the 5-cycle which contains no clique of three vertices and no stable set of three vertices. Hence $r(3,3) \geq 6$. By Note 12.A and Theorem 12.9 we have $r(3) \leq r(3,2) + r(2,3) = 3 + 3 = 6$, and therefore $r(3,3) = 6$. In Figure 12.11(b), we have a graph that contains no clique on three vertices and not stable set on four vertices (this is the "Wagner graph" of Exercise 10.5.14 and Figure 10.23). hence $r(3,4) \geq 9$. By Note 12.A, $r(2,4) = 4$ and so by Theorem 12.9 (the second claim; $r(2,4)$ and $r(3,3)$ are both even) $r(3,4) \leq r(3,3)+r(2,4) = 1 = 6+4-1 = 9$, and therefore $r(3,4) = 9$. In Figure 12.11(c) we have a graph that contains (we claim; this is harder to see but the symmetry of the graph helps) no clique on three vertices and no stable set on five vertices. Hence $r(3,5) \geq 14$. By Note 12.A, $r(2,5) = 5$ and so by Theorem 12.9 $r(3,5) \leq r(3,4) + r(2,5) = 9 + 5 = 14$, and therefore $r(3,5) = 14$. In Figure 12.11(d) we have a graph that contains (again, not easy to see) no clique on four vertices and no stable set on four vertices. Hence $r(4,4) \geq 18$. Since $r(3,4) = r(4,3) = 9$ then by Theorem 12.9 $r(4,4) \leq r(4,3) + r(3,4) = 9 + 9 = 18$, and therefore $r(4,4) = 18$. Bondy and Murty give the following table of all known Ramsey numbers $r(k,\ell)$ where we take $3 \leq k \leq \ell$ (without loss of generality, because $r(k,\ell) = r(\ell,k)$ and, by Note 12.1, $r(1,\ell) = r(k,1) = 1$ and $r(2,\ell) = r(\ell,2) = \ell$):

k	3	3	3	3	3	3	3	4	4
ℓ	3	4	5	6	7	8	9	4	5
$r(k,\ell)$	6^*	9^*	14^*	18	23	28	36	18^*	25

We have established the values marked with an asterisk. A document giving the latest results on these studies is "Small Ramsey Numbers" document by Stanisław P. Radziszowski (Revision #16: January 15, 2021; accessed 5/7/2022). The Wikipedia webpage on Ramsey's theorem includes some of the same information; it is less academic, but might be updated faster in the event of new discoveries (accessed 5/7/2022).

Definition. A (k, ℓ)-*Ramsey graph* is a graph on $r(k, \ell) - 1$ vertices that contains neither a clique on k vertices nor a stable set on ℓ vertices.

Note. A (k, ℓ)-Ramsey graph established a lower bound on the Ramsey number $r(k, \ell)$. Each of the graphs in Figure 12.11 are Ramsey graphs. Figures 12.11(1), (b), (c), (d) give, respectively, a $(3, 3)$-Ramsey graph, a $(3, 4)$-Ramsey graph, a $(3, 5)$-Ramsey graph,and a $(4, 4)$-Ramsey graph.

Note. The $(3, 5)$-Ramsey graph of Figure 12.11(c) can be associated with the finite field $\langle \mathbb{Z}_{13}, +, \cdot \rangle$ as follows. The vertices of the graph are labeled with the elements of $\mathbb{Z}_{13}$ and two vertices are adjacent if and only if the difference of their labels is a cube modulo 13; namely 1, 5, 8, or 12. Similarly, the $(4, 4)$-Ramsey graph of Figure 12.11(d) can be associated with the field $\langle \mathbb{Z}_{17}, +, \cdot \rangle$ by labeling the vertices with the elements of $\mathbb{Z}_{17}$ and joining two vertices if and only if the difference of their labels is a square modulo 17, namely 1, 2, 4, , 8,9, 13, 15, or 16.

Note. In fact, each graph in Figure 12.11 is a circulant graph (that is, a Cayley graph based on a cyclic group $\mathbb{Z}_n$; see Exercises 1.3.18 and 1.3.19). The $(3, 3)$-Ramsey graph of Figure 12.11(a) is the circulant graph $CG(\mathbb{Z}_5, \{1, 4\})$, the $(3, 4)$-Ramsey graph of Figure 12.11(b) is the circulant graph $CG(\mathbb{Z}_8, \{1, 4, 7\})$, the $(3, 5)$-Ramsey graph of Figure 12.11(c) is the circulant graph $CG(\mathbb{Z}_{13}, \{1, 5, 8, 12\})$, and the $(4, 4)$-Ramsey graph of Figure 12.11(d) is the circulant graph $CG(\mathbb{Z}_{17}, \{1, 2, 4, 8, 9, 13, 15, 16\})$.

Note. Now we put upper bounds on $r(k, \ell)$ and lower bounds on $r(k, k)$.

Theorem 12.10. For all positive integers k and ℓ, $r(k, \ell) \leq \binom{k+\ell-2}{k-1}$.

Note. The upper bound of $\binom{k+\ell-2}{k-1}$ on $r(, k, \ell)$ of Theorem 12.10 represents the number of subsets of size $k-1$ of a set of size $k+\ell-2$. Since there are $2^{K+\ell-2}$ subsets (of any size) of a set of size $K+\ell-2$ (that is, for $|A| = n$, the power set $\mathcal{P}(!)$ satisfies $|\mathcal{P}(A)| = 2^n$). Hence $\binom{k+\ell-2}{k-1} \leq 2^{k+\ell-2}$ so that we can take this as an upper bound on $r(k, \ell)$, as follows.

Corollary 12.11. For all positive integers k and ℓ, $r(k, \ell) \leq 2^{k+\ell-2}$, with equality if and only if $k = \ell = 1$.

Note. The bound given on $r(k, \ell)$ in Corollary 12.11 is weaker than that given in Theorem 12.10, but we consider it to show that the diagonal Ramsey numbers $r(k, k)$ grow at most exponentially. In the next theorem we give a lower bound on $r(k, k)$ that also grows exponentially.

Theorem 12.12. For all positive integers k, $r(k, k) \geq 2^{k/2}$.

Note. As commented in Note 12.B, we can interpret $r(k, \ell)$ as the minimum value of n such that every 2-edge-coloring of K_n contains either a monochromatic K_k or a monochromatic K_ℓ. With this approach, we can generalize the idea of a Ramsey number from two colors to several colors.

Definition. For positive integers t_i, $1 \le i \le k$, define the *general Ramsey number* $r(t_1, t_2, \ldots, t_k)$ is the smallest integer n such that every k-edge-coloring $(E_1, E_2 m \ldots, E_k)$ (where E_i is the set of edges which are color i) of K_n contain a complete subgraph on t_i vertices all of whose edges belong to E_i for some $1 \le i \le k$. (Notice that the *generalized* Ramsey number is defined in Exercise 12.3.7 based on monochromatic subgraphs that may not be complete graphs.)

Note. The following theorem and corollary (proofs of which are to be given in Exercise 12.3.3) give generalizations of Theorems 12.9 and 12.10.

Theorem 12.13. For all positive integers t_i, $1 \le i \le k$,

$$r(t_1, t_2, \ldots, t_k) \le r(t_1 - 1, t_2, t_3, \ldots, t_k) + r(t_1, t_2 - 1, t_3, \ldots, t_k)$$

$$+ r(t_1, t_2, t_3 - 1, t_4, \ldots, t_k) + \cdots + r(t_1, t_2, \ldots, t_{k-1}, t_k - 1) - k + 2.$$

Corollary 12.14. For all positive integers t_i, $1 \le i \le k$,

$$r(t_1 + 1, t_2 + 1, \ldots, t_k + 1) \le \frac{(t_1 + t_2 + \cdots + t_k)!}{t_1! t_2! \cdots t_k!}.$$

Note. We conclude this section by considering partitioning sets of integers. Notice that

$$\{1, 2, \ldots, 13\} = \{1, 4, 10, 13\} \cup \{2, 3, 11, 12\} \cup \{5, 6, 7, 8, 9\}.$$

Notice that we cannot choose x, y, z from the same set on the right-hand-side in such a way that $x + y = z$. This does not hold for all such sets; for example, no matter how we partition $\{1, 2, \ldots, 14\}$ into three subsets, one of these subsets will contain x, y, z such that $x+y = z$. In 1916, Axel Schur proved that for any positive integer n (the number of subsets), there exists a positive integer r_n (the size of the set of integers to be partitioned) $\{1, 2, \ldots, r_n\}$ into n subsets, at least one of the subsets contains x, y, z with $x + y = z$. In fact, we can take r_n to be the general Ramsey number $r(t_1, t_2, \ldots, t_n)$ where $t_i = 3$ for $1 \le i \le n$, as we now show.

Theorem 12.15. Schur's Theorem.
Let $\{A_1, A_2, \ldots, A_n\}$ be a partition of the set of integers $\{1, 2, \ldots, r_n\}$ into n subsets. Then some A_i contains three integers x, y, and z satisfying the equation z.

Revised: 5/21/2022