Section 13.4. Evolution of Random Graphs

Note. In this section, we define a monotone property of a graph and give conditions under which certain examples of these properties almost certainly hold or almost certainly do not hold.

Definition. If $\mathbf{P}$ is an *monotone* property of graphs (i.e., a property which is preserved when edges are added), a *threshold function* for $\mathbf{P}$ is a function $f(n)$ such that:

1. if $p \ll f(n)$, then $G \in \mathcal{G}_{n,p}$ almost surely does not have property $\mathbf{P}$,

2. if $p \gg f(n)$, then $G \in \mathcal{G}_{n,p}$ almost surely has property $\mathbf{P}$.

Note. An example of a monotone property is the property $\mathbf{P}$ that graph $G \in \mathcal{G}_{n,p}$ contains a triangle, since adding edges to G preserves the property that G has a triangle. We saw in Note 13.2.B that if $pn \to \infty$ as $n \to \infty$ then $\mathbf{P}$ almost surely does not hold. In Note 13.3.A we saw that if $pn \to \infty$ as $n \to \infty$ then property $\mathbf{P}$ almost surely does hold. Therefore $f(n) = n^{-1}$ is a threshold function for $\mathbf{P}$. Notice that we say “a” threshold function, since (for example) any positive constant multiple of n^{-1} would also be a threshold function (with the constant being absorbed in the limit process. It is to be shown in Exercise 13.4.1 that every monotone property of graphs has a threshold function.

Definition. Graph F is *balanced* if the average degrees of the proper subgraphs of F do not exceed the average degree $d(F) = 2e(F)/v(F)$ of the graph F itself.

Note. The class of balanced graphs includes trees and regular graphs (and so includes cycles and complete graphs). The next result concerning the threshold function for the property **P** of graph G containing balanced graph F as a subgraph. It appears in Paul Erdös and A. Rényi's "pioneering" paper "On the Evolution of Random Graphs," *Magyar Tudomanyos Akademia Matematikai Kutato Intezetenek Kozlmenenyei* **5**, 17–61 (1960). A copy of this work (which is in English) is available online at Cite Seer X (accessed 12/23/2020).

Theorem 13.11. Let F be a nonempty balanced graph with k vertices and l edges. Then $n^{-k/l}$ is a threshold function for the property of containing F as a subgraph.

Note. As an example of an application of Theorem 13.11, let F be a tree on k vertices and let property **P** be the monotone property that G has F as a subgraph. Then F has $l = k - 1$ edges (by Theorem 4.3) and so by Theorem 13.11 $f(n) = n^{-k/l} = n^{-k/(k-1)}$ is a threshold function for G to contain such a tree. The number of nonisomorphic trees on k vertices is less than k^{k-2} (since this is the number of labeled trees on k vertices by Cayley's Theorem, Theorem 4.8), so the number of nonisomorphic trees on k vertices is less than k^{k-2}. So when $p \ll f(n) = n^{-k/(k-1)}$, G almost surely does not have property **P** (that is, G almost surely has no subgraph which is a tree on k vertices, and so G has no connected component on k vertices [since such a component would have a spanning subtree by Theorem 4.6]). When $p \gg n^{-k/(k-1)}$, G almost surely has property **P** (that is, G almost surely has a connected component on k or more vertices). Now the threshold function for the property of G having a cycle as a subgraph (for cycles $k = l$) by Theorem 13.11 is $g(n) = n^{-k/l} = n^{-1} > n^{-k/(k-1)} = f(n)$. So for $1/n > p \gg n^{-k/(k-1)}$ we have that G is almost surely acyclic and so is almost surely a forest, with at least one component of size at least k.

Note. A related application of Theorem 13.11 is given in Exercise 13.4.2 where it is to be shown that for $p = (\log n - f(n))/n$, where $f(n) \to \infty$ as $n \to \infty$, we have that G almost surely has no isolated vertices.

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