Section 13.5. The Local Lemma

Note. In this section, we state and prove The Local Lemma which concerns a collection of events for which the intersection of the events has positive probability. We then use it to address two colourable hypergraphs, even cycles in digraphs, and "linear arboricity."

Note 13.5.A. We can use the probabilistic method to show the existence of certain graph properties. For example, if we are interested in a proper k-colouring of graph G, we could consider randomly colouring the vertices of G with k colours and then calculate the probability that the colouring is proper. If this probability is nonzero then G must be k-colourable. Now we would have a k-colouring if for each edge e in G, the ends of e receive a different colour. let A_e be the event that the ends of e are assigned the *same* colour (this is a "bad" event). We want the complement events $\overline{A}_e$ to occur for all edges e of G. So we are interested in $P(\cap_{e\in E}\overline{A}_e)$. If we can show this probability is positive then G must be k-colourable.

Note. In general, if $\{A_i\}_{i=1}^n$ is a set of "bad" events in a finite probability spce (Ω, P) then we are interested in $P\left(\cap_{i\in N}\overline{A}_i\right)$ where $N = \{1, 2, \ldots, n\}$ If the events A_i are independent and each occurs with probability strictly less than one, then by Exercise 13.2.2

$$P\left(\cap_{i\in N}\overline{A}_i\right) = \prod_{i\in N} P(\overline{A}_i) = \prod_{i\in N}(1 - P(a_i)) > 0,$$

as "desired."

Note. It is more common that a set of events are not independent. For example if edges e, f, g of G are edges of a triangle then, as is shown in Exercise 13.1.1,

$$P(A_e \cap A_f \cap A_g) = 1/k^2 > 1/k^3 = P(A_1)P(A_f)P(A_g).$$

But Erdös and Lovász (..) showed that $P\left(\cap_{i\in N}\overline{A}_i\right)$ will be positive if the events A_i occur with low probability an dare "to be a sufficient extent" independent of one another. Details are given below in The Local Lemma.

Definition. If $\{A_i \mid i \in S\}$ is a set of events in a finite probability space, then we denote as A_S the intersection $A_S = \cap_i \in SA_i$. An event A_i is *independent* of a set of events $\{A_j \mid j \in J\}$ if for all subsets $S \subseteq J$ we have $P(A_i \cap A_S) = P(A_i)P(A_S)$.

Theorem 13.12. The Local Lemma.

Let A_i, where $i \in N$, be events in a finite probability space (Ω, P) and let $N_i \subseteq N$ where $i \in N$. Suppose that, for all $i \in N$,

(i) A_i is independent of the set of events $\{A_j \mid J \in N_i\}$,

(ii) for each $i \in N$, there is a constant p_i where $0 < p_i < 1$, and for each $i \in N$ we have $P(A_i) = p_i \prod_{j\in N_i}(1-p_j)$.

Set $B_i = \overline{A}_i$ where $i \in N$. Then, for any two disjoint subsets $R, S \subseteq N$,

$$P(B_R \cap B_S) \geq P(B_R)\prod_{i\in S}(1-p_i). \qquad (13.15)$$

In particular, when $R = \varnothing$ and $S = N$,

$$P\left(\cap_{i\in N}\overline{A}_i\right) \geq \prod_{i\in N}(1-p_i) > 0. \qquad (13.16)$$

Note. In The Local Lemma, if the A_i are mutually independent then the p_i defined in point (ii) are probabilities (or more precisely, they *can* be probabilities; other values can be used because of all the inequalities) and $P(A_i) = p_i$ (so that $P(\overline{A}_i) = 1 - p_i$ and then the non-strict inequality in (13.16) becomes an equality). When the A_i are not independent, the value p_i associated with A_i is reduced by the "compensation factor" $\prod_{j \in N_i}(1 - p_j)$ (using the term of Bondy and Murty).

Note. In our applications, we don't need the full power of The Local Lemma but instead we need a special case. The special case requires the following definition.

Definition. Consider the events A_i, where $i \in N$, in a finite probability space, and the subsets N_i of N such that A_i is independent of $\{A_j \mid j \notin N_i\}$. Form the (strict or "simple") digraph D with vertex set N and arc set $\{(i, j) \mid i \in N, j \in N_i\}$. Such a digraph is a *dependence digraph*; if it is symmetric then it is called a *dependency graph* (in which we replace two oppositely oriented arcs by an edge, as is a standard of Bondy ad Murty; see page 33).

Note. In the example in Note 13.5.A where the vertices of graph G are randomly assigned one of k colours, we denoted as A_e the event that the ends of e are assigned the same colour. Then event A_e is independent of $\{A_f \mid f \text{ is nonadjacent to } e\}$. So in the dependency graph for the events A_e where $e \in G$, the vertex set in $E(G)$ and two vertices e and f are adjacent in the dependency graph is edges e and f are adjacent in G. This is exactly the line graph of G. Bondy and Murty comment: "In general, there are many possible choices of dependency digraph (or graph) for a given set of events. . . " (page 357).

Theorem 13.14. THE LOCAL LEMMA—SYMMETRIC VERSION.
let A_i, where $i \in N$, be events in a finite probability space (Ω, P) having a dependency graph with maximum degree d. Suppose $P(A_i) < 1/(e(d+1))$ for all $i \in N$ (where "e" here is the base of the natural log function). Then $P(\cap_{i \in N} \overline{A}_i) > 0$.

Note. We now give applications of the symmetric version of The Local Lemma to three areas: two colourable hypergraphs, even cycles in digraphs, and "linear arboricity" (to be defined below).

Theorem 13.15. Let $H = (V, \mathcal{F})$ be a hypergraph in which each edge has at least k elements and meets at most d other edges. If $e(d+1) \le 2^{k-1}$ (again, "e" here is the base of the natural log function), then H is 2-colourable.

Corollary 13.16. Let $H = (V, \mathcal{F})$ be a k-uniform k-regular hypergraph, where $k \ge 9$. Then H is 2-colourable.

Note. The next result, due to N. Alon and N. Linial in 1989, uses The Local Lemma to prove that all diregular digraphs of sufficiently high degree have directed cycles of even lengths.

Theorem 13.17. Let D be a strict (i.e., "simple") k-diregular digraph where $k \ge 8$. Then D contains a directed even cycle.

Definition. A *linear forest* in a graph $G = (V, E)$ is a subgraph of G, each component of which is a path. When G is decomposed into as few as possible linear forests, the number of linear forests is the *linear arboricity* of G, denoted $\mathrm{la}(G)$.

Note 13.5.B. Every graph has a decomposition into linear forests since a single edge is a linear forest. If $G = K_{2n}$, then the linear arboricity is equal to n, because K_{2n} admits a decomposition into Hamilton paths by Exercise 2.4.6. For an arbitrary graph G, we'll see in Chapter 17 ("Edge Colourings") that the linear arboricity is bounded above by the "edge chromatic number" (see Section 17.1. Edge Chromatic Number), the minimum number of 1-factors (see Section 16.4. Perfect Matchings and Factors) into which the graph can be decomposed. We'll see in Vizing's Theorem (Theorem 17.4) that the edge chromatic number is bounded above by $\Delta + 1$, so we have for G that $\mathrm{la}(G) \leq \Delta + 1$. For a lower bound on the linear arboricity we can just count edges. For example, if G is $2r$-regular then $m = (2rn)/2 = rn$ and, because no linear forest has more than $n - 1$ edges (namely, when the linear forest is a Hamilton path), then $\mathrm{la}(G) \geq \left\lceil \frac{rn}{n-1} \right\rceil = \left\lceil r\frac{n}{n-1} \right\rceil = r + 1$. We'll show below that in fact $\mathrm{la}(G) = r + 1$ for $2r$-regular graphs of sufficiently large girth. We need the next lemma (the proof of which uses The Local Lemma).

Lemma 13.18. Let $G = (V, E)$ be a simple graph and let $\{V_1, V_2, \ldots, V_k\}$ be a partition of V into k sets, each of cardinality at least $2e\Delta$ (again, "e" here is the base of the natural log function). Then there is a stable set S in G such that $|S \cap V_i| = 1$ for $1 \leq i \leq k$.

Note. Recall from Section 2.1 ("Subgraphs and Supergraphs") that the *girth* of a graph G is the length of the shortest cycle in G. Recall form Section 2.2 ("Spanning and Induced Subgraphs") that a k-factor of a graph is a spanning k-regular subgraph. We now use Lemma 13.18 (which is based on The Local Lemma) to show the bound on the linear arboricity of a $2r$-regular graph given in Note 13.5.B above is in fact sharp (i.e., the inequality reduces to equality).

Theorem 13.19. Let $G = (V, E)$ be a simple $2r$-regular graph with girth at least $2e(4r - 2)$ (again, "e" here is the base of the natural log function). Then $\text{la}(G) = r + 1$.

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