Chapter 14. Vertex Colourings

Section 14.1. Chromatic Number

Note. In Section 11.1. Colourings of Planar Maps we introduced vertex colourings in the context of the Four-Colour Conjecture. In this section we state (or restate) the definitions we need to address vertex colourings. We state a heuristic to find a colouring of a given graph (this will require us to recall some of the algorithms and results stated in Section 6.1. Tree Search). We also consider colourings of digraphs.

Definition. A *k-vertex colouring* (or simply *k-colouring*) of graph $G = (V, E)$ is a mapping $c : V \to S$, where S is a set of k *colours.* A colouring c is *proper* if no two adjacent vertices are assigned the same colour. A graph is *k-colourable* if it has a proper k-vertex colouring.

Note. When we consider edge colourings in Chapter 17, we include the description "edge," so when you hear "coluring" it indicates a vertex colouring. We usually denote the set of colours as $\{1, 2, \ldots, k\}$. Notice that a graph with loops cannot be properly coloured.

Definition. For k-vertex colouring $c : V \to S$ where $S = \{1, 2, \ldots, k\}$ is the set of colours, the set $V_i = \{v \in V \mid c(v) = i\}$ is the ith *colour class* of the colouring.

Note 14.1.A. A k-colouring of a graph is equivalent to a partitioning $\{V_1, V_2, \ldots, V_k\}$ of the vertex set V where V_i are colour classes. If the k-colouring is proper, then each colour class is a stable set (i.e., an "independent set") in the graph. In this chapter, we are only interested in proper colourings. **In this chapter we refer to a proper colouring as just a "colouring" and a proper k-colouring as a "k-colouring."** Notice then that a graph is 2-colourable if and only if it is bipartite (and then the partite sets determine the colour classes and conversely). A loopless graph is k-colourable if and only if its underlying simple graph is k-colourable. **So in this chapter we restrict our attention to simple graphs.**

Definition. The minimum k for which graph G is k-colourable is the *chromatic number* of G, denoted $\chi(G)$. If $\chi(G) = k$ then G is *k-chromatic.*

Note. Every odd cycle is 3-colourable. Every even cycle is 2-colourable (because even cycles are bipartite graphs). The complete graph K_n is n-chromatic. Recall that the stability number of graph G, $\alpha(G)$, is the cardinality of a maximum stable set in G. Since a stable set corresponds to a colour class in a k-colouring by Note 14.1.A, then

$$\chi(G) \geq n/\alpha(G) \tag{14.1}$$

because the number of stable sets in G is at least $n/\alpha(G)$.

Note. Colourings of graphs are useful in applications requiring certain partitions, as the next example illustrates.

Example 14.1. Examination Scheduling. The students at a small university have finals in all the courses they take. Exams in different courses cannot be held at the same time if the courses have students in common. How can all the exams be organized in as few parallel sessions as possible? To find such a schedule, consider the graph G whose vertex set is the set of all courses, with two courses being joined by an edge if they give rise to a conflict. Stable sets of G correspond to conflict-free groups of courses. Therefore the required minimum number of parallel sessions is the chromatic number of G.

Note 14.1.B. If graph G is k-colourable and H is a subgraph of G then H is certainly k-colourable. So, in terms of chromatic numbers, $\chi(G) \geq \chi(H)$. If G contains some clique K_r then $\chi(G) \geq \chi(K_r) = r$. So in terms of clique number ω (recall that $\omega(G)$ is the maximum size, that is maximum number of vertices, of a clique of graph G) we have $\chi \geq \omega$. This inequality can be strict, as illustrated by odd length cycles of length at least 5, since these graphs have $\omega = 2$ and $\chi = 3$.

Note. A polynomial-time algorithm exists for determining whether or not a graph is bipartite (see Exercise 6.1.3 where the Breadth-First Search polynomial-time algorithm is to be modified to test for bipartite-ness). But, according to Bondy and Murty, the problem of determining if a graph is 3-colourable is $\mathcal{NP}$-complete, so that the problem of finding the chromatic number of a graph is $\mathcal{NP}$-hard (see Chapter 8, "Complexity of Algorithms," for a discussion of this terminology). So computationally, we settle for an approximation of the chromatic number. Recall from Section 8.5, "Greedy Heuristics," Bondy and Murty state: "A *heuristic* is a computational procedure, generally based on some simple rule, which intuition tells one should usually yield a good approximate solution to the problem at hand. . . . a *greedy heuristic* is a procedure which selects the best current option at each stage, without regard to future consequences." We wouldn't expect a greedy heuristic to yield an optimal result, but as we saw in Section 8.5 with the Borůvka-Kruskal Algorithm (Algorithm 8.22) and Theorem 8.23, sometimes it does produce an optimal result (a minimal weight spanning tree in a weighted graph, in the Borůvka-Kruskal Algorithm). We now state a greedy heuristic to get a colouring of a graph.

Heuristic 14.3. THE GREEDY COLOURING HEURISTIC.

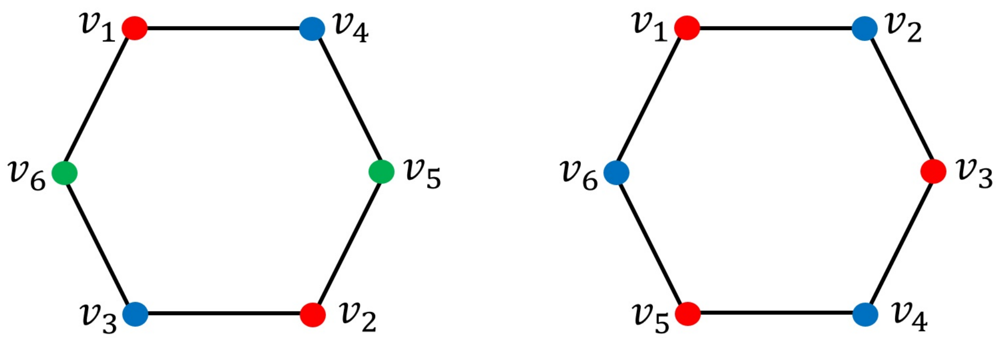
INPUT: a graph G

OUTPUT: a colouring of G

1. Arrange the vertices of G in a linear order: $v_1, v_2, \ldots, v_n$.

2. Colour the vertices one by one in this order, assigning to v_i the smallest positive integer not yet assigned to one of its already-coloured neighbors.

Note. The number of colours needed in the Greedy Colouring Heuristic can depend on the linear ordering of the vertices. For example if $G = C_6$ with the linear order given in the figure below (left) then 3 colours are needed by the Greedy Heuristic. Applying Step 2 gives (using actual colour names, ordered as red, blue, green, instead of numbers) v_1 is red, v_2 is red, v_3 is blue, v_4 is blue, v_5 is green, and v_6 is green. If we use the linear order given in the figure below (right) then 2 colours are needed by the Greedy Heuristic (not surprising, since C_6 is bipartite).



However, it is to be shown in Exercise 14.1.9 that for any graph G, there is an ordering of its vertices such that the greedy heuristic yields a colouring with $\chi(G)$ colours. Though this observation is not useful to find $\chi(G)$ since one does not know the linear ordering in advance.

Lemma 14.1.A. The number of colours needed in the Greedy Colouring Heuristic (Heuristic 14.3) is at most $\Delta + 1$. Therefore, for any graph G, $\chi(G) \le \Delta + 1$.

Note. The bound $\Delta + 1$ on $\chi(G)$ of Lemma 14.1.A is best possible (or "sharp"), as seen when G is an odd length cycle or a complete graph. In fact, these are the only graphs for which the inequality in Lemma 14.1.A reduces to equality. This was shown by R.L. Brooks in "On Colouring the Nodes of a Network," *Mathematical Proceedings of the Cambridge Philosophical Society*, **37**(2), 194–197 (1941). Before stating and proving the result, recall that the Depth-First Search algorithm (Algorithm 6.4) produces a rooted spanning tree for any connected graph. In Exercise 6.1.11 it is to be shown that for connected graph G, if every DFS-tree is a Hamilton path, then G is either a cycle, a complete graph, or a complete bipartite graph in which both partite sets are the same size (i.e., $G = C_{|V|}$, $G = K_{|V|}$, or $G = K_{|V|/2,|V|/2}$).

Theorem 14.4. Brooks' Theorem.
If G is a connected graph, and is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.

Definition. A (proper) *vertex colouring* of digraph D is a vertex colouring of its underlying graph G, and its *chromatic number*, $\chi(D)$, is the chromatic number $\chi(G)$ of G.

Note. The above definition may lead one to think that vertex colourings of digraphs tells us little that we didn't already know from vertex colourings of graphs. However, the next result shows there are "directed properties" related to digraph vertex colourings. This result from the late 1960s is due to T. Gallai and B. Roy.

Theorem 14.5. The Gallai-Roy Theorem.
Every digraph D contains a directed path with $\chi(D)$ vertices.

Note. The remainder of the notes for this section are optional. In fact, the material does not appear in some printings of Bondy and Murty.

Note. Recall that the Gallai-Milgram Theorem (Theorem 12.2) states that for any digraph D, $\pi(D) \leq \alpha(D)$, where $\pi(D)$ is the least number of directed paths in a path partition of D and $\alpha(D)$ is the largest stable set (or "independent set") of the underlying graph of D. This beats "a striking formal resemblance" (as Bondy and Murty put it on page 366) to the Gallai-Roy Theorem (Theorem 14.5). In a (proper) colouring, each colour class is a stable set (see Note 14.1.A). By interchanging the roles of directed paths and stable sets (or colour classes), one theorem is transformed into the other; path partitions become colourings (that is, partitions of the vertex set into colour classes). However, this discussion does not address the orthogonality of directed paths and stable sets (recall from Section 12.1. Stable Sets that directed path P and stable set S are orthogonal if they have one common vertex). An attempt to find a common generalization of the Gallai-Milgram Theorem and the Gallai-Roy Theorem was conjectured by N. Linial in 1981 (see Exercise 14.1.22). We next discuss a related conjecture, after stating some necessary definitions.

Definition. Let k be a positive integer. A path partition $\mathcal{P}$ is *k-optimal* if it minimizes the function $\sum\{\min\{v(P), k\} \mid P \in \mathcal{P}\}$, and a *partial k-colouring* of a graph or digraph is a family of k disjoint stable sets. A (directed) path partition $\mathcal{P}$ and partial k-colouring $\mathcal{C}$ are *orthogonal* if every directed path $P \in \mathcal{P}$ meets $\min\{v(P), k\}$ different colour classes of $\mathcal{C}$.

Note. Notice that a 1-optimal path partition is a path partition that is optimal. A partial 1-colouring is a stable set. In 1982, C. Berge stated the following conjecture concerning orthogonality, k-optimal path partitions, and partial k-colourings.

Conjecture 14.6. The Path Partition Conjecture.

Let D be a digraph, k a positive integer, and $\mathcal{P}$ a k-optimal path partition of D. Then there is a partial k colouring of D which is orthogonal to $\mathcal{P}$.

Note. The Path Partition Conjecture has been proved for $k = 1$ and $k = 2$. In Exercise 14.1.22, it is to be shown that Linial's Conjecture mentioned above is implied by the Path Partition Conjecture.

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