Section 14.2. Critical Graphs

Note. In this brief section, we consider graphs that are minimal k-colourable graphs. We give several definitions are prove a few properties of critical graphs.

Definition. A graph G is *colour-critical* (or simply *critical*) if $\chi(H) < \chi(G)$ for every proper subgraph H of G. A k*-critical* graph is one that is k-chromatic and critical.

Note. Of course every k-chromatic graph has a k-critical subgraph. The Grözsch graph of Figure 14.4 is 4-critical. We next give some properties of critical graphs.

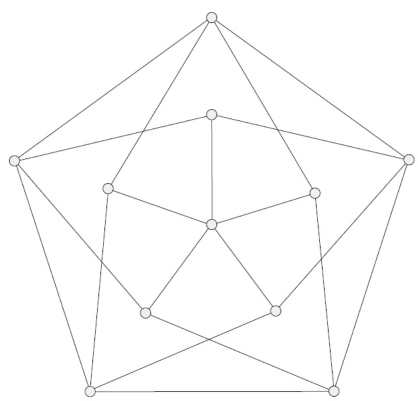

Fig. 14.4. The Grötzsch graph: a 4-critical graph

Theorem 14.7. If G is k-critical then $\delta \geq k - 1$.

Definition. Let S be a vertex cut of a connected graph G and let the connected components of $G - S$ have vertex sets $V_1, V_2, \ldots, V_t$. The subgraphs $G_i = G[V_i \cup S]$

are the S-*components* of G (as defined in Section 9.4. Three-Connected Graphs). Colourings of $G_1, G_2, \ldots, G_t$ are said to *agree* on S if, for every vertex $v \in S$, vertex v is assigned the same colour in each of the colourings.

Theorem 14.8. No critical graph has a clique cut.

Note. Recall that we are considering simple, loopless graphs in this chapter. So a separating vertex and a cut vertex are the same (see Note 5.2.A). So by Theorem 14.8 no critical graph has a separating vertex. This gives the following.

Corollary 14.9. Every critical graph is nonseparable.

Note. Theorem 4.8 implies that if a k-critical graph has a 2-vertex cut set $\{u, v\}$, then u and v cannot form a clique; that is, u and v cannot be adjacent. So in a proper colouring, the S-components (where $S = \{u, v\}$) of G may or may not have u and v of the same colour.

Definition. If G is a k-critical graph with a 2-vertex cut $\{u, v\}$, then a $\{u, v\}$-component G_i of G is of *Type 1* if every $(k-1)$-colouring of G_i assigns the same colour to u and v, and of *Type 2* if every $(k-1)$-colouring of G_i assigns distinct colours to u and v.

Note. Figure 14.5 gives the $\{u, v\}$-components of the 4-chromatic Hajór graph. Notice that the $\{u, v\}$-component on the left is Type 1 (u and v must be the same colours in a 3-colouring) and the $\{u, v\}$-component on the right is Type 2 (u and v must be different colours in a 3-colouring). We use the Type 1/Type 2 categorization of $\{u, v\}$-components in the proof of the next theorem. It is due to G. A. Dirac and appears in "The Structure of k-Chromatic Graphs," *Fundamenta Mathematicae*, **40**, 42-55 (1953). There are links to online copies of this work on the Semantic Scholar website.

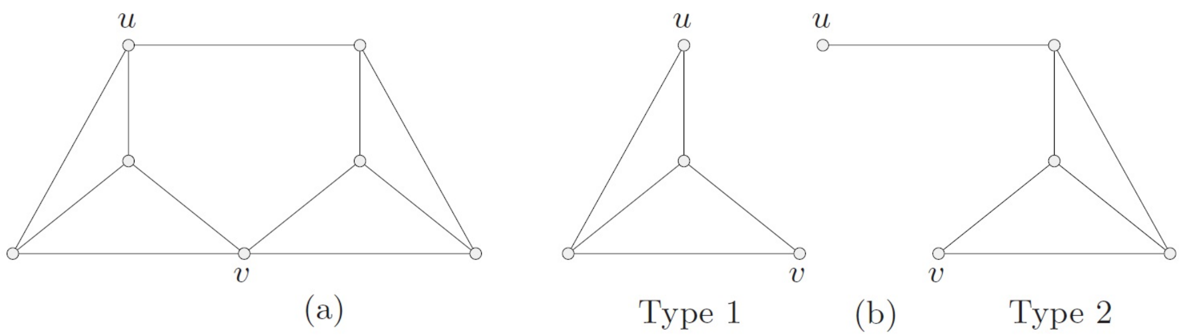

Fig. 14.5. (a) A 2-vertex cut $\{u, v\}$ of the Hajós graph, (b) its two $\{u, v\}$-components

Theorem 14.10. Let G be a k-critical graph with a 2-vertex cut set $\{u, v\}$, and let e be a new edge joining u and v. Then

(1) $G = G_1 \cup G_2$, where G_i is a $\{u, v\}$-component of G of Type i for $i \in \{1, 2\}$,

(2) both $H_1 = G_1 + e$ and $H_2 = G_2/\{u, v\}$ are k critical.