Section 14.3. Girth and Chromatic Number

Note. A graph with a large clique must have an (at least equally) large chromatic number (see Note 14.1.B). So if a graph is triangle-free (triangles are small cliques), we might conjecture that the chromatic number is also "small." In this section we show (in Theorem 14.12) that for any $k \in \mathbb{N}$, there is a k-chromatic triangle-free graph.

Note. Recall that the girth of a graph with at least one cycle is the length of the shortest cycle (see Section 2.1. Subgraphs and Supergraphs). The next result might also seem surprising since large girth means no cliques of size larger than two, and yet the chromatic number is also alarge. The technique of proof is the probabilistic method and is due to Paul Erdős in "Graph Theory and Probability. II," *Canadian Journal of Mathematics*, **13**, 346–352 (1961). A copy of Erdős' paper is available online on the Cambridge.org website (accessed 6/13/2022).

Theorem 14.11. For each positive integer k, there exists a graph with girth at least k and chromatic number at least k.

Note. In Theorem 14.11, the existence of a graph of girth at least k and chromatic number at least k is guaranteed. The graph may not have exactly the given value k as wither the girth of the chromatic number, but *at least* this value (as given in the probabilistic method of proof). Also, the proof is not constructive and we have no idea, based on the proof, how to find such a graph. Recursive constructions of such graphs were given by László Lovász in "On Chromatic Number of Finite Set-Systems," *Acta Mathematica Academiae Scientiarum Hungaricae*," **19**(1–2), 59–67 (1968) and by Jaroslav Nešetřil and Vojtěch Rödl in "A Short Proof of the Existence of Highly Chromatic Hypergraphs without Short Cycles," *Journal of Combinatorial Theory, Series B*, **27**, 225–227 (1979). A copy of the first source is online on Lovász's webpage and a copy of the second can be found on Science Direct. The next theorem has a constructive proof and addresses the case of graphs of girth greater than three (i.e., trinagle-free graphs). This theorem is due to Jan Mycielski and appears in "Sur le Coloriage des Graph," *Colloquium Mathematicae*, **3**(2), 161–162 (1955). A copy can be accessed online through the European Digital Mathematics Library. These three websites were accessed 6/13/2022.

Theorem 14.12. For any positive integer k, there exists a triangle-free k-chromatic graph.

Note. In Exercise 14.3.2 "shift graphs" are used to give graphs with arbitrarily high chromatic numbers which are triangle-free (in fact, they are of girth at least six).

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