Section 14.4. Perfect Graphs

Note. In this section define perfect graphs, imperfect graphs, and minimally imperfect graphs. We state and prove the Perfect Graph Theorem which related the "perfection" of a graph to the perfection of its complement. Finally, we state the Strong Perfect Graph Theorem which classifies perfect graphs; we motivate the result, but do not give a proof.

Note. In Note 14.1.B, we observed that the chromatic number $\chi(G)$ and the clique number $\omega(G)$ (the number of vertices of a largest clique in G) are related as $\chi(G) \geq \omega(G)$. If we are interested in the graphs in which this reduces to equality, one can play a trick. Any k-chromatic graph H can be used to create a graph G where $\chi(G) = \omega(G)$ by simply defining $G = H \cup K_k$ where H and K_k are disjoint (though G is then not connected). Claude Berge in "Some Classes of Perfect Graphs," in *Six Papers in Graph Theory*, Indian Statistical Institute, Calcutta, 1–21 (1963), decided to impose the equality $\chi = \omega$, not only on graph G, but also on all induced subgraphs of G. He called such graphs perfect graphs.

Definition. A graph is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. Otherwise, it is *imperfect*. An imperfect graph is *minimally imperfect* if each of its proper induced subgraphs is perfect.

Note. In his 1963 paper, Berge proved that several families of graphs are perfect, including bipartute graphs, line graphs of bipartite graphs (see Section 1.3. Graphs Arising from Other Structures) and chordal graphs (see Section 9.7. Chordal Graphs). The exercises of this section relate to showing certain graps are perfect or minimally imperfect (Exercise 14.4.2 claims that odd-length cycles of length five or more, and their complements, are minimally imperfect).

Note. Notice that it is easy to see that a bipartite graph is perfect. Every induced subgraph is bipartite. A bipartite graph with no edges has $\chi = \omega = 1$, and a bipartite graph with at least one edge has $\chi = \omega = 2$.

Note. Berge in his 1963 paper conjectured that the complement of a perfect graph is perfect. This was prove by L. Lovász in "Normal Hypergraphs and the Perfect Graph Conjecture," *Discrete Mathematics*, **2**, 253–267 (1972). A copy is online on the Science Direct website (accessed 6/15/2022). It is now known as the Perfect Graph Theorem and we will justify it below in Note 14.4.A.

Theorem 14.13. The Perfect Graph Theorem.
A graph is perfect if and only if its complement is perfect.

Note. The next theorem gives a classification of perfect graphs in terms of the stability number and the clique number.

Theorem 14.14. A graph G is perfect if and only if every induced subgraph H of G satisfies the inequality $v(H) \leq \alpha(H)\omega(H)$.

Note 14.4.A. With H as a graph and $\overline{H}$ as its complement, because a stable set of H determines a clique of $\overline{H}$ (and a clique of H determines a stable set of $\overline{H}$) we have $v(\overline{H}) = v(H)$, $\alpha(\overline{H}) = \omega(H)$, and $\omega(\overline{H}) = \alpha(H)$. Therefore, Theorem 14.14 implies that a graph is perfect if and only if its complement is perfect. That is, Theorem 14.14 implies the Perfect Graph Theorem (Theorem 14.13). We need two preliminary results before probing Theorem 14.14. Both deal with minimally imperfect graphs.

Proposition 14.15. Let S be a stable set in a minimally imperfect graph G. Then $\omega(G - S) = \omega(G)$.

Lemma 14.16. Let G be a minimally imperfect graph with stability number α and clique number ω. Then G contains $\alpha\omega+1$ stable sets $S_0, S_1, \ldots, S_{\alpha\omega}$ and $\alpha\omega+1$ cliques $C_0, C_1, \ldots, C_{\alpha\omega}$ such that:

- each vertex of G belongs to precisely α of the stable sets S_i,
- each clique C_i has ω vertices,
- $C_i \cap S_i = \varnothing$ for $0 \leq i \leq \alpha\omega$, and
- $|C_i \cap S_j| = 1$ for $0 \leq i < j \leq \alpha\omega$.

Note. By Exercise 14.4.2, odd length cycles of length five or more, as well as their complements, are minimally imperfect. So $\overline{C_7}$ as given in Figure 14.7(b) is minimally imperfect.

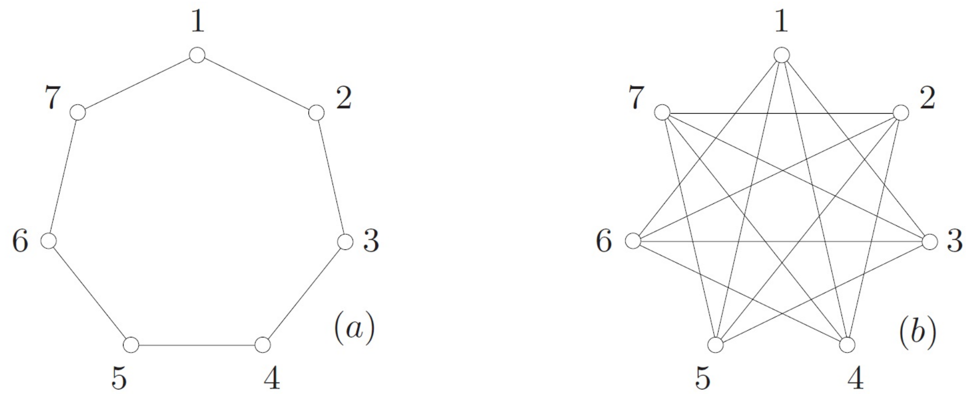

Fig. 14.7. The minimally imperfect graphs (a) C_7, and (b) $\overline{C_7}$

To illustrate the proof of Lemma 14.6, we consider this graph. We have $\alpha(\overline{C_7}) = 2$ and $\omega(\overline{C_7}) = 3$ ($\overline{C_7}$ contains several triangles by no K_4). Applying the technique used in the proof of Lemma 14.16, we have the following $\alpha\omega + 1 = 7$ stable sets and cliques:

$S_0 = \{1, 2\}$	$S_1 = \{2, 3\}$	$S_2 = \{4, 5\}$	$S_3 = \{6, 7\}$
$C_0 = \{3, 5, 7\}$	$C_1 = \{1, 4, 6\}$	$C_2 = \{1, 3, 6\}$	$C_3 = \{1, 3, 5\}$
	$S_4 = \{3, 4\}$	$S_5 = \{5, 6\}$	$S_6 = \{1, 7\}$
	$C_4 = \{2, 5, 7\}$	$C_5 = \{2, 4, 7\}$	$C_6 = \{2, 4, 6\}$

Here, $\mathcal{S}_1 = \{S_1, S_2, S_3\}$ and $\mathcal{S}_2 = \{S_4, S_5, S_6\}$. The proof of Lemma 14.16 does not give insight on how to find the cliques, but $\overline{C_7}$ is small enough that given two vertices in a stable set, there is a unique clique (a triangle, since $\omega = 3$) disjoint from the stable set. For example, with $S_0 = \{1, 2\}$, we see that the only triangle on the subgraph of $\overline{C_7}$ induced by vertices $3, 4, 5, 6, 7$ is the triangle with vertex set $\{3, 5, 7\}$. The incidence matrices $\mathbf{S}$ and $\mathbf{C}$ for the S_i and C_i are given in Figure

14.8. We use such matrices now in the proof of Theorem 14.14.

S :

	S_0	S_1	S_2	S_3	S_4	S_5	S_6
1	1	0	0	0	0	0	1
2	1	1	0	0	0	0	0
3	0	1	0	0	1	0	0
4	0	0	1	0	1	0	0
5	0	0	1	0	0	1	0
6	0	0	0	1	0	1	0
7	0	0	0	1	0	0	1

C :

	C_0	C_1	C_2	C_3	C_4	C_5	C_6
1	0	1	1	1	0	0	0
2	0	0	0	0	1	1	1
3	1	0	1	1	0	0	0
4	0	1	0	0	0	1	1
5	1	0	0	1	1	0	0
6	0	1	1	0	0	0	1
7	1	0	0	0	1	1	0

Fig. 14.8. Incidence matrices of families of stable sets and cliques of $\overline{C_7}$

Note. In Exercise 14.4.6 it is to be shown that the next corollary follows from the Perfect Graph Theorem.

Corollary 14.17. A graph G is perfect if and only if, for any induced subgraph H of G, the maximum number of vertices in a stable set of H is equal to the minimum number of cliques required to cover all the vertices of H.

Note. If a graph is perfect then, by definition, all of its induced subgraphs are perfect. So we can classify perfect graphs by describing all minimally imperfect graphs. As mentioned above, Exercise 14.4.2 it is to be shown that odd length cycles of length five or more and the complements of such graphs are minimally imperfect. Claude Berge conjectured in his 1963 paper that these are the only minimally imperfect graphs. This is equivalent to saying that a graph is perfect if and only if it contains no odd cycle of length at least five and it contains no complement of such a cycle. This became known as the Strong Perfect Graph Conjecture. It was proved in M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, "The Strong Perfect Graph Theorem," *Annals of Mathematics*, **164**(2), 51–229 (2006). A copy of the paper can be viewed online on the *Annals of Mathematics* website (accessed 6/17/2022). Notice the length of the paper.

Theorem 14.19. The Strong Perfect Graph Theorem.
A graph is perfect if and only if it contains no odd cycle of length at least five, or the complement of such a cycle, as an induced subgraph.

Note. A polynomial-time recognition algorithm for perfect graphs was given in M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour, and K. Vušković, "Recognizing Berge Graphs," *Combinatorica*, **25**, 143–186 (2005). A copy is available online on the CiteSeer^X website (accessed 6/17/2022).

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