Section 14.5. List Colourings

Note. In a list colouring the vertices of a graph are assigned colours, but the colour assigned to a particular vertex must be chosen from a list associated with the vertex. In this way, some colours are disallowed from being assigned to certain vertices. In applications, this related to putting vertices/items into certain categories (i.e., colour classes) wjere some items cannot be put into some categories.

Definition. Let G be a graph and let L be a function which assigns to each vertex v of G a set $L(v)$ of positive integers, called the *list* (of colours) of v. A colouring $c : V \to \mathbb{N}$ such that $c(v) \in L(v)$ for all $v \in V$ is a *list colouring* of G with respect to L, or an *L-colouring*, and G is said to be *L-colourable.*

Note. If $L - \{1, 2, \ldots, k\}$ for all $v \in V$, then an L-colouring is equivalent to a k-colouring. If a vertex is assigned a list consisting of a single colour, then this amounts to precolouring the vertex with that colour. As with "regular" vertex colourings, the idea of list colourings can be extended to digraphs by simply considering list colourings of the underlying graph.

Note. Surprisingly, a graph may be k-chromatic, yet have a list $L(v)$ of length k associated with each vertex, yet the graph may not be L-colourable. Consider the graph and lists of Figure 14.9. The graph is bipartite and so is 2-chromatic. Yet the list associated with each vertex is of length 2 but the lists do not admit an L-colouring of the graph (consider the two possible colours of the upper-left vertex and trace through the graph to get a contradiction). However, it is to be shown in Exercise 14.5.1 that is each list is of length three, then an L-colouring exists. This example motivates the following definition.

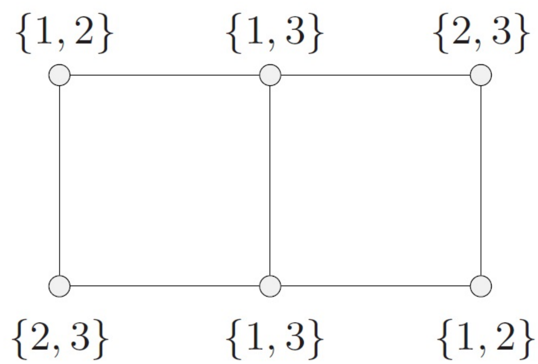

Fig. 14.9. A bipartite graph whose list chromatic number is three

Definition. A graph G or digraph D is *k-list-colourable* if it has a list colouring whenever all the lists have length k. The smallest value of k for which G is k-list-colourable is the *list chromatic number* of G, denoted $\chi_L(G)$.

Note. For graph G of Figure 14.9, we have $\chi(G) = 2$ and $\chi_L(G) = 3$. In Exercise 14.5.5, it is to be shown that there exists 2-chromatic graphs (in fact, bipartite) of arbitrarily large list chromatic number.

Note. Recall that, from Section 12.1. Stable Sets, a *kernel* in a digraph D is a stable set S of D such that each vertex of $D - S$ dominates some vertex of S (where "u dominates v" means that (u, v) is an arc of D). The next theorem relates kernels of a digraph to L-colourings.

Theorem 14.20. Let $D = (V, A)$ be a digraph each of whose induced sub-digraphs has a kernel. For $v \in V$, let $L(v)$ be an arbitrary list of at least $d^+(v) + 1$ colours. Then D admits an L-colouring.

Note. By Richardson's Theorem (Theorem 12.6), a digraph which does not contain an odd-length directed cycle has a kernel (and hence every induced subgraph has a kernel since the induced subgraphs do not contain directed odd-length cycles). So Theorem 14.20 can be applied to this class of graphs to give the next corollary. Now every graph has an orientation without a directed cycle; this is obtained by numbering the vertices 1 to n and orienting every edge from the vertex with the lesser number to the vertex with the larger label. This lets us also deduce a result concerning list-colourable graphs.

Corollary 14.21. Let $D = (V, A)$ be a digraph which contains no directed odd-length cycle. For $n \in V$, let $L(v)$ be an arbitrary list of at least $d^+(v) + 1$ colours. Then D admits an L-colouring. In particular, every graph G is $(\Delta + 1)$-list-colourable.

Note. In Exercise 14.5.10, it is to be shown that every interval graph G (see Section 1.3. Graphs Arising from Other Structures) has an acyclic orientation D (i.e., D contains no directed cycles) with $\Delta^+ \leq \omega - 1$. Combining this with Theorem 14.20 gives the following.

Corollary 14.22. Every interval graph G has list chromatic number ω.