Section 14.6. The Adjacency Polynomial

Note. In this section we introduce a multivariate polynomial based on the edges of a graph (i.e., based on the adjacency structure). We illustrate its connections to proper colourings and orientation, and use it to show the existence of list colourings.

Definition. Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Set $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. The *adjacency polynomial* of G is the multivariate polynomial function

$$A(G, \mathbf{x}) = \prod_{i<j} \{(x_i - x_j) \mid v_i v_j \in E\}.$$

This is sometimes called simply the *graph polynomial*.

Note 14.6.A. Let $\mathbb{F}$ be a field and $\mathbf{c} \in \mathbb{F}^n$ where $\mathbf{c} \neq \mathbf{0}$. Then $\mathbf{c}$ can be regarded as a function $c : V \to \mathbb{F}$ where $c(v_i) = c_i$ (where c_i is the ith component of $\mathbf{c}$). with the elements of $\mathbb{F}$ as colours, c yields a proper colouring of G is and only if $x_i \neq x_j$ for each $v_i v_y \in E$. That is, $A(G, \mathbf{c}) \neq 0$ if and only if $c : V \to \mathbb{F}$ is a proper colouring of G. Recall that a field has no zero divisors; see my online notes for Introduction to Modern Algebra (MATH 4127/5127) on Section IV.19. Integral Domains, in particular the definition of integral domain (Definition 19.6) and the fact that every field is an integral domain (Theorem 19.9). This is why we cannot use a more general algebraic structure for the colours.

Note. Since the number of edges of G is m, then $A(G, \mathbf{x})$ consists of 2^m monomial terms in the expansion before simplification (since there are 2 choices, x_i and x_j, in each of the m terms $(x_i - x_j)$). Orienting the edge v_iv_j by making v_i the tail, we have each of the 2^m orientations of G from the 2^m terms of $A(G, \mathbf{x})$. For example, if G is the graph of Figure 14.10 (left) then its adjacency polynomial is

$$A(G, \mathbf{x}) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4).$$

The $2^5 = 32$ terms, after simplification gives

$$\begin{aligned} A(G, \mathbf{x}) \;=\; & x_1^3x_2x_3 - x_1^3x_2x_4 - x_1^3x_3^2 + x_1^3x_3x_4 - x_1^2x_2^2x_3 + x_1^2x_2^2x_4 \\ & -x_1^2x_2x_3x_4 + x_1^2x_2x_4^2 + x_1^2x_3^3 - x_1^2x_3x_4^2 + x_1x_2^2x_3^2 - x_1x_2^2x_4^2 \\ & -x_1x_2x_3^3 + x_1x_2x_3^2x_4 - x_1x_3^3x_4 + x_1x_3^2x_4^2 - x_2^2x_3^2x_4 + x_2^2x_3x_4^2 \\ & +x_2x_3^3x_4 - x_2x_3^2x_4^2 + x_2x_3x_4^3 - x_2x_4^4 - x_3^2x_4^3 + x_3x_4^4. \end{aligned}$$

Notice that a monomial term of he form $\pm x_1^2x_2x_3x_4$ results from the three products $x_1x_1(-x_4)x_2x_3$, $x_1(-x_3)x_1x_2(-x_4)$, and $(-x_2)x_1x_2(-x_3)(-x_4)$. Each gives an outdegree sequence for $v_1, v_{2,3}, v_4$ of $(2, 1, 1, 1)$ and there are three orientations of G with this outdegree sequence, as given in Figure 14.10. Notice that when the three terms of the form $\pm x_1^2x_2x_3x_4$ are added, this results in the single term $-x_1^2x_2x_3x_4$ in the simplified version of $A(G, \mathbf{x})$ given above.

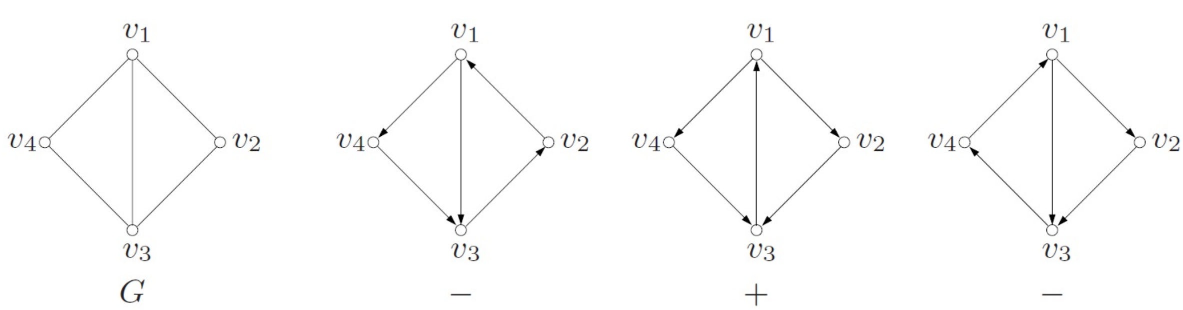

Fig. 14.10. A labelled graph G and the three orientations corresponding to the term $x_1^2x_2x_3x_4$ of its adjacency polynomial

The signs given in Figure 14.10 are explained below.

Note/Definition. In the complete graph K_n, every pair of vertices are adjacent so the adjacency polynomial is $A(K_n, \mathbf{x}) = \prod_{1 \le i < j \le n} (x_i - x_j)$. This multivariate polynomial is the determinant of the *Vandermonde matrix* (or the *Vandermonde determinant*):

$$A(K + n, \mathbf{x}) = \prod_{1 \le i < j \le n} (x_i - x_j) = \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{vmatrix}.$$

This equality is to be established in Exercise 8.11 of James Gentle's *Matrix Algebra: Theory, Computations, and Applications in Statistics*, 2nd Edition, Spring (2017) where, as is common, the transpose of out Vandermonde matrix is considered; this is the book used in ETSU's Theory of Matrices (MATH 5090), though when teaching it I do not cover Chapter 8 "Special Matrices and Operations Useful in Modelling and Data Analysis" (see my online notes for Theory of Matrices). In Bondy and Murty's Exercise 4.6.1 it is to be shown that the number of monomial terms in the Vandermonde determinant above is $n!$ (s that many terms must cancel in the $2^{\binom{n}{2}}$ monomials in the expansion of the adjacency polynomial).

Definition. Let D be an orientation of graph G. The *sign* of the orientation is

$$\sigma(D) = \prod\{\sigma(e) \mid a \in A(D)\} \text{ where}$$

$$\sigma(a) = \begin{cases} +1 & \text{if } a = (v_i, v_j) \text{ with } i < j \\ -1 & \text{if } a = (v_i, v_j) \text{ with } i > j. \end{cases}$$

Note. For example, the arcs of the third orientation D of G in Figure 14.10 has arcs (a_1, v_2), (v_1, v_3), (v_2, v_3), (v_3, v_4), and (v_4, v_1). Now $\sigma(v_1v_2) = \sigma(v_1v_3) = \sigma(v_2v_3) = +1$ and $\sigma(v_4v_2) = -1$ so that $\sigma(D) = (+1)^3(-1) = -1$.

Definition. Let G be a graph with n vertices and m edges. Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a sequence of nonnegative integers whose sum is m. The *weight* of $\mathbf{d}$ is $w(\mathbf{d}) = \sum \sigma D$ where the sum is taken over all orientations D of G whose outdegree sequence is $\mathbf{d}$.

Note. We denote $\mathbf{x}^{\mathbf{d}} = \prod_{i=1}^{n} x_i^{d_i}$ and then the adjacency polynomial is of the form $A(G, \mathbf{x}) = \sum_{\mathbf{d}} e(\mathbf{d})\mathbf{x}^{\mathbf{d}}$. This is the case because $\mathbf{x}^{\mathbf{d}}$ is simply the "forms" of the monomials in $A(G, \mathbf{x})$ and $w(\mathbf{d})$ is the coefficient of the monomial (a sum of +1's and −1's) after simplification.

Note. The "Hilbert Nullstellensatz" (or Zeros Theorem of Hilbert) is a result of classical algebraic geometry. It is stated and proved in Chapter VIII, "Commutative Rings and Modules," Section VIII.7., "The Hilbert Nullstellensatz," of

Thomas Hungerford's *Algebra*, Graduate Texts in Mathematics #73, Springer-Verlag (1974). This is the book I use in the ETSU Modern Algebra sequence (MATH 5410, MATH 5420). See my online notes for this sequence on Rings and Modules (though I do not cover Chapter VIII). For the sake of comparison with the "Combinatorial Nullstellensatz," we observe that Hungerford's statement of the Nullstellensatz is as follows.

Proposition VIII.7.4. Hilbert Nullstellensatz.

Let F be an algebraically closed extension field of field K and I a proper ideal of $K[x_1, x_2, \ldots, x_n]$. Let

$$V(I) = \{(a_1, a_2, \ldots, a_n \in F^n \mid g(a_1, a_2, \ldots, a_n) = 0 \text{ for all } g \in I\}.$$

Then

$$\text{Rad}\, I = J(V(I)) = f \in K[x_1, x_2, \ldots, x_n] \mid f(a_1, a_2, \ldots, a_n) = 0$$

$$\text{for all } (a_1, a_2, \ldots, a_n \in V(I)\}.$$

In other words, $f(a_1, a_2, \ldots, a_n) = 0$ for every zero $(a_1, a_2, \ldots, a_n)$ of I in F^n if and only if $f^m \in I$ for some $m \geq 1$.

According to the 2022 manuscript Kriti Goel, Dilip Patil, and Jugal Verma's "Nullstellensätze and Applications" (available online on arxiv.org; accessed 6/20/2022), David Hilbert proved the result in five pages of the third section of his paper on invariant theory. The reference is: David Hilbert, "Über die vollen Invariantensysteme [On Full Invariant Systems]," *Mathematische Annalen*, **42**, 313–373 (1893). A copy (in German) is online on the European Digital Mathematics Library (accessed 6/21/2022).

Note. The Combinatorial Nullstellensatz also considers a multivariate polynomial over a field. The zeros of the polynomial are related to list colourings. It was proved by Noga Alon in "Combinatorial Nullstellensatz," *Combinatorics, Probability and Computing*, **8**, 7–29. A copy is available online on the University of California, Davis webpage of Jesús A. De Loera (accessed 6/20/2022). Before stating the Combinatorial Nullstellensatz, we need a preliminary result concerning zeros of a polynomial over a field.

Proposition 14.23. Let f be a polynomial, not the zero polynomial, over a field $\mathbb{F}$ in the variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, of degree d_i in x_i for $1 \leq i \leq n$. Let L_i be a set of $d_i + 1$ elements of $\mathbb{F}$ for $1 \leq i \leq n$. Then there exists $\mathbf{t} \in L_1 \times L_2 \times \cdots \times L_n$ such that $f(\mathbf{t}) \neq 0$.

Note. We now have the equipment to prove the Combinatorial Nullstellensatz.

Theorem 14.24. The Combinatorial Nullstellensatz.
Let f be a polynomial over a field $\mathbb{F}$ in the variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Suppose that the total degree of f is $\sum_{i=1}^n d_i$ and that the coefficient in f of $\prod_{i=1}^n x_i^{d_i}$ is nonzero. Let L_i be a set of $d_i + 1$ elements of $\mathbb{F}$ for $1 \leq i \leq n$. Then there exists $\mathbf{t} \in L_1 \times L_2 \times \cdots \times L_n$ such that $f(\mathbf{t}) \neq 0$.

Note. By Note 14.6.A, the existence of $\mathbf{t} \in L_1 \times L_2 \times \cdots \times L_n$ such that $f(\mathbf{t}) \neq 0$ where f is the adjacency polynomial $A(G, \mathbf{x})$ implies a list colouring of G. So Proposition 14.23 and the Combinatorial Nullstellensatz (Theorem 14.24) can ve used to explore list colourings.

Corollary 14.25. If G has an odd number of orientations D with outdegree sequence $\mathbf{d}$, then G is $(\mathbf{d} + \mathbf{1})$-list-colourable.

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