Section 14.7. The Chromatic Polynomial

Note. In this section we introduce another polynomial (this time, one of a single variable) associated with a graph. This polynomial gives the number of k-colourings of G and is hence called the chromatic polynomial of G. We present a recursive technique for the computation of chromatic polynomials. *In this section we allow loops and parallel edges*, unless stated otherwise.

Note. George D. Birkhoff (March 21, 1884–November 12, 1944) studied the number of k-colourings of a graph when considering the Four-Colour Conjecture in "A Determinant Formula for the Number of Ways of Coloring a Map," *Annals of Mathematics*, **14**(1/4), 42–46 (1912/13). A copy is available online on the JSTOR website (no password is required; accessed 6/22/2022). We denote the number of distinct k-colourings $c : V \to \{1, 2, \ldots, k\}$ of a graph G by $C(G, k)$. So $C(G, k) > 0$ if and only if G is k-colourable, and if G has a loop then $C(G, k) = 0$. To count k-colourings, we need a way to distinguish between different colourings. An example of the computation of the number of colourings of an elementary map is given in the supplement Supplement. The Four-Color Theorem: A History, Part 2; see Note FCT.R.

Definition. Two colourings of graph G are *distinct* if some vertex is assigned different colours in the two colourings. That is, if $\{V_1, V_2, \ldots, V_k\}$ and $\{V_1', V_2', \ldots, V_k'\}$ are two k-colourings (where the V_i and V_i' are colour classes) then the *k-colourings are equal*, $\{V_1, v_2, \ldots, V_k\} = \{V_1', V_2', \ldots, V_k'\}$, if and only if $V_i = V_i'$ for $1 \leq i \leq k$.

Note. Consider a triangle on vertices v_1, v_2, v_3. There are six distinct 3-colourings: (1) $V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}$, (2) $V_1 = \{v_1\}, V_2 = \{v_3\}, V_3 = \{v_2\}$, (3) $V_1 = \{v_2\}, V_2 = \{v_1\}, V_3 = \{v_3\}$, (4) $V_1 = \{v_2\}, V_2 = \{v_3\}, V_3 = \{v_1\}$, (5) $V_1 = \{v_3\}, V_2 = \{v_1\}, V_3 = \{v_2\}$, and (6) $V_1 = \{v_3\}, V_2 = \{v_2\}, V_3 = \{v_1\}$. So $C(K_3, 3) = 6$. Similarly, there are $n!$ n-colourings of K_n. Of course these colourings are "structurally" very similar and we could define some type of isomorphism or equivalence classes of k-colourings, but instead we follow our definition of "distinct" in the counting process and the value of $C(G, k)$.

Note 14.7.A. If G has n vertices and no edges, then each vertex can be assigned any of the k colours so that $C(G, k) = k^n$. In a k-colouring of $G = K_n$ (where $k \geq n$), the first vertex v can be assigned any of the k colours, the second vertex v_2 can be assigned any of the remaining $k-1$ colours, and so forth so that $G(K_n, k) = C(G, k) = k(k-1)\cdots(k-n+1)$ (the number of permutations of k objects taken n at a time). We are using the Fundamental Counting Principle here, and we will also require it in the exercises; see my online notes for Applied Combinatorics and Problem Solving (MATH 3340) on Section 1.1. The Fundamental Counting Principle.

Note. In Exercise 14.7.1 it is to be shown that the $C(G, k)$ satisfies the recursion formula:

$$C(G, k) = C(G \setminus e, k) - C(G/e, k). \tag{14.5}$$

Note. Notice that the recursion formula (14.5) is similar in structure to the formula for the number of spanning trees $t(G)$ of G as given in Proposition 4.9: $t(g) = t(G \setminus e) + t(G/e)$. Recursion formula (14.5) yields the following recursion formula for the chromatic polynomial. In the following, it is claimed that the coefficients "alternate in sign," which means that the coefficients alternate as nonpositive, nonnegative, nonpositive, nonnegative, etc. That is, we interpret "alternate in sign' be allowing 0 to be considered as either positive or negative, as needed.

Theorem 14.26. For any loopless graph G, there exists a polynomial $P(G,x)$ such that $P(G,x) = C(G,k)$ for all nonnegative integers k. Moreover, if G is simple and e is any edge of G, then $P(G,x)$ satisfies the recursion formula:

$$C(G,x) = C(G \setminus e, x) - C(G/e, x). \tag{14.6}$$

The polynomial $P(G,x)$ is of degree n, with integer coefficients which alternate in sign, leading term x^n, and constant term 0.

Definition. The polynomial $P(G,x)$ of Theorem 14.26 which gives the number of k-colourings of loopless graph G is the *chromatic polynomial* of G.

Note. Recursion formula (14.6) gives us two ways to find $P(G,x)$:

(i) We can start with graph G and repeatedly delete and contract edges and use the formula $P(G,x) = P(G \setminus x) - P(G/e,x)$ to express $P(G,x)$ as an integer linear combination of chromatic polynomials of empty graphs.

(ii) We can start with a complete graph and repeatedly delete and contract edges (a complete graph in which an edge is contracted yields a complete graph on one fewer vertices, but with multiple edges, but multiple edges do not affect the number of k-colourings) and use the formula $P(G \setminus e, x) = P(G, x) + P(G/e, x)$ to express $P(G, x)$ as an integer linear combination of chromatic polynomials of complete graphs.

By Note 14.7.A, we know $C(G, k)$ (and hence $P(G, x)$) for empty graphs and complete graphs. The first method is best for finding chromatic polynomials for graphs with few edges, whereas the second method is best for finding chromatic polynomials for graphs with "many" edges (that is, graphs that are "close to" complete graphs). Both techniques are to be used in Exercise 14.7.2.

Note. In Section 14.1. Chromatic Number, in order to motivate the Greedy Colouring Heuristic, we observed that finding the chromatic number of a graph is $\mathcal{NP}$-hard. If a polynomial-time algorithm were known for finding the chromatic polynomial of a graph, then this could be used to find the chromatic number of a graph (also in polynomial-time). So no such algorithm is known. Hence, finding the chromatic polynomial for a given graph often involves expressing it in terms of chromatic polynomials of subgraphs. This is illustrated in some of the exercises (see Exercises 14.7.6(a), 14.7.7, and 14.7.8).

Note. The question of which polynomials are chromatic polynomials remains unanswered. In Exercise 14.7.3(b) it is to be shown that $x^4 - 3x^3 + 3x^2$ is *not* a chromatic polynomial (but notice that it does satisfy the properties of a chromatic polynomial as given in Theorem 14.26).

Definition. Roots of a chromatic polynomial are called *chromatic roots*.

Note 14.7.B. Using recurrence relation (14.6) one can show that 0 is the only real chromatic root less than 1, as is to be done in Exercise 14.7.9. Thinking in terms of the number of k-colourings of a graph, we see that 0 is a root of every chromatic root for nonempty (that is, graphs with at least one edge), loopless graphs since such graphs do not have 1-colourings.

Note. In the study of roots of polynomials one should, in the humble opinion of your instructor, do so in the context of complex roots (in this way, we have the power of the Fundamental Theorem of Algebra with us). Alan D. Sokal proved in "Chromatic Roots are Dense in the Whole Complex Plane," *Combinatorics, Probability and Computing*, **13**, 221–261 (2004) that, as the title states, the set of chromatic roots of (all) chromatic polynomials form a dense subset of $\mathbb{C}$. Recall that one set is *dense* in another (in some topological space) if the closure of the dense set equals the other set. So what Sokal has shown is that for any complex number z_0, for all $\varepsilon > 0$ there is a chromatic root in $\{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}$. This is Sokal's Corollary 1.3; the paper can be viewed online on the University

College London website (accessed 6/24/2022). This might be surprising in light of the fact that the only real chromatic root less than 1 is 0 (see Note 14.7.B). This simply means that there are complex numbers which are chromatic roots which are arbitrarily close to all real numbers less than 1 (or arbitrarily close to any real number…or for that matter, arbitrarily close to any complex number!).

Note. As a passing observation, we note that Alan Sokal is mentioned in my online presentation Introduction to Math Philosophy and Meaning (originally prepared for the Department of Biological Sciences class "Great Ideas in Science 1" [BIOL 3018]). In the 1990s he became a critic of "radical postmodernism" (my term). He published an intentionally nonsensical paper in a well-respected cultural studies journal which involved applications of quantum gravity to social settings and a claim that there is no physically real world but that it is merely a social and linguistic construct. He tells his story in his book with Jean Bricmont, *Fashionable Nonsense: Postmodern Intellectual's Abuse of Science*, Picador (1999).

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