Chapter 15. Colourings of Maps

Note. In this chapter we outline Appel, Hakens, and Kock's proof of the Four Colour Theorem (and also explore Kempes' incorrect proof of the 1880s). We consider list colourings of planar graphs and extend the idea of map colourings to surfaces other than the plane and sphere.

Section 15.1. Chromatic Numbers of Surfaces

Note. In this section, we define the chromatic number of a closed surface Σ, denoted $\chi(\Sigma)$, and put upper bounds on this number for some surfaces. We show by example that this upper bound is attained for a torus. The basic idea of this section is the extend the Four Colour Theorem from the plane to other surfaces.

Note. The Four Colour Theorem (Theorem 11.2) implies that every planar graph has chromatic number at most four. In Theorem 10.4 we claimed that a graph is planar (or "embeddable in the plane") if and only if it is embeddable in the sphere (this idea is justified based on stereographic projection between the extended plane and the sphere). Hence, every graph embeddable in the sphere has chromatic number at most four. We make the transition from the plane to the sphere in order to have a conversation about surfaces. Recall from Section 10.6. Surface Embeddings of Graphs that a surface (or "2-manifold") is *closed* if it is bounded and has no boundary.

Theorem 15.1.A. For every closed surface Σ, there is a least integer k such that every graph embeddable on Σ has chromatic number at most k.

Definition. For a closed surface Σ, the least integer k such that every graph embeddable on Σ is k-colourable is the *chromatic number* of Σ, denoted $\chi(\Sigma)$.

Note. Notice that Theorem 15.1.A does not *give* $\chi(\Sigma)$, but only gives an upper bound on $\chi(\Sigma)$. Figure 15.1 lists four closed surfaces, their Euler number $c(\Sigma)$, and upper bounds on their chromatic number $\chi(\Sigma)$.

Σ	$c(\Sigma)$	$\chi(\Sigma)$
sphere	2	≤ 6
projective plane	1	≤ 6
torus	0	≤ 7
Klein bottle	0	≤ 7

Fig. 15.1. Bounds on the chromatic numbers of various surfaces

Note. We can refine the bound on $\chi(\Sigma)$ given in the proof of Theorem 15.1.A when $c(\Sigma) < 0$ with the following bound given by Percy J. Heawood in "Map-Colour Theorem," *Quarterly Journal of Pure and Applied Mathematics*, **24**, 332–338 (1890). This is partially reprinted in N. L. Biggs, E. K. Lloyd, R. J. Wilson's *Graph Theory: 1736-1936*, (NY: Clarendon Press/Oxford, 1976); see pages 105–107. It was in this paper that Heawood pointed out an error in Altred Kempe's (then 10-year-old) "proof" of the Four Colour Theorem (to be discussed in the next section).

Theorem 15.1. Heawood's Inequality.

For any closed surface Σ with Euler characteristic $c \leq 0$ we have

$$\chi(\Sigma) \leq \frac{1}{2}\left(7 + \sqrt{49 - 24c}\right).$$

Note 15.1.A. Since we only have upper bounds on $\chi(\Sigma)$, we can get equality if we can find a specific graph with chromatic number equal to an upper bound on $\chi(\Sigma)$. For the torus, chromatic number is at most 7, as stated in Figure 15.1. Of course $\chi(K_7) = 7$ and an embedding of K_7 on the torus (originally due to Heawood) is given in Figure 3.9(a), so 7 is the chromatic number of the torus.

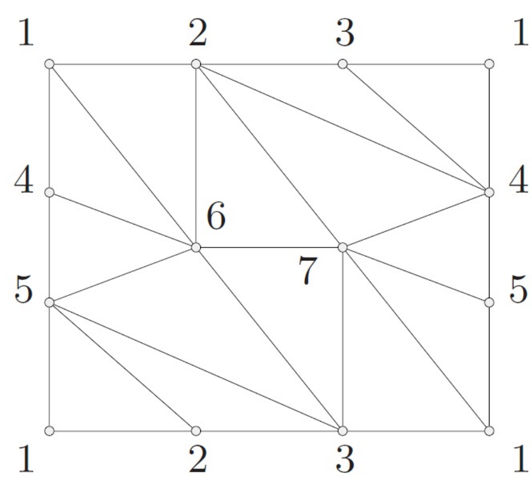

Fig. 3.9. (a) Toroidal embeddings of the complete graph K_7

In the other direction, we can show that the upper bound of $\chi(\Sigma)$ is not in fact the least upper bound by example. From Figure 15.1 we see that for Σ as the Klein bottle we have $\chi(\Sigma) \leq 7$. Phillip Franklin showed in "A Six-Colour Problem," *Journal of Physics*, **13**, 363–369 (1934), that K_6 can be embedded in the Klein bottle (and of course it is 6-colourable). It's dual (called the *Franklin graph*) partitions the Klein bottle into 6 regions (one for each vertex), giving a map on the Klein bottle that is 6-face colourable. In Figure 15.2, the faces of the map are given in (b) and the dual of it, the 6-colourable graph on 6 vertices, is given in (a).

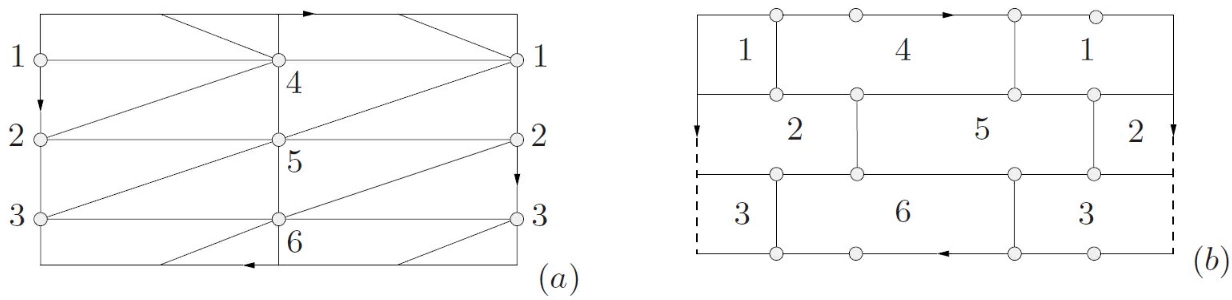

Fig. 15.2. (a) A 6-chromatic triangulation of the Klein bottle, and (b) its dual, the Franklin graph

The Franklin graph is given in Figure 15.3 with three crossings (which coincide in this drawing).

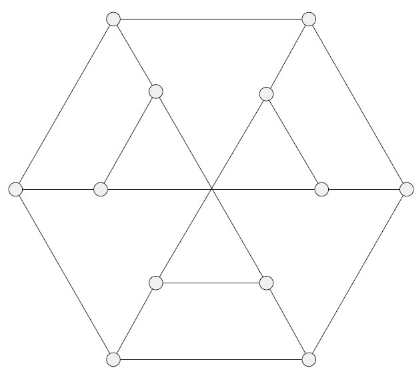

Fig. 15.3. Another drawing of the Franklin graph

Note. Heawood conjectured that equality holds in Theorem 15.1 for every surface Σ of Euler characteristic $\chi(\Sigma) \leq 0$. This became known as the Heawood Map-Colouring Conjecture. However, as argued in Note 15.1.A, the torus has chromatic number at most 6, whereas Theorem 15.1 implies that the chromatic number is at most 7 (because the Euler characteristic of the torus is 0). It turns out that this is the only exception to Heawood's conjecture. It was shown in Gerhard Ringel and J.W.T Youngs' "Solution of the Heawood Map-Colouring Problem," *Proceedings of the National Academy of Sciences, U.S.A*, **60**, 438–445 (1968) (available online on the Proceedings of the National Academy of Sciences webpage; accessed 7/18/2022) that Heawood's Map-Coluring Conjecture holds for all surfaces of Euler characteristic at most 0, with the exception of the torus. This result is now known as the Map Colour Theorem.

Note. As for surfaces of positive characteristic, there are only the projective plane (of characteristic 1) and the plane or sphere (of characteristic 2). The Four Colour Theorem shows that the chromatic number of the sphere is 4 (which, in fact, also agrees with the bound given by Theorem 15.1, even though it is only necessarily true for surfaces of nonpositive characteristic). We have a bound of 6 on the chromatic number of the projective plane, as given in Figure 15.1. In Figure 25(a) we have an embedding of K_6 on the projective plane, establishing 6 as the chromatic value of the projective plane (interestingly, this also agrees with the bound given by Theorem 15.1 if we were to plug $c = 1$ into the bound).

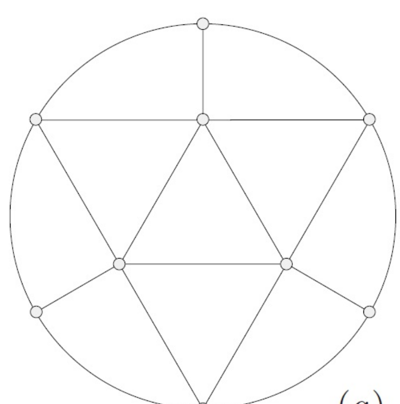

Fig. 10.25. (a) Embeddings on the projective plane of K_6

We have only considered closed surfaces, but the ideas of this section can be extended to surfaces with boundaries. In Exercise 15.1.1 it is to be shown by example that the chromatic number of the Möbius strip is at least six.

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