Section 15.2. The Four-Colour Theorem

Note. We stated the Four-Colour Theorem in Section 11.1 as:

Theorem 11.2. The Four-Colour Theorem.

Every plane graph without cut edges is 4-face colourable.

A proof was given by Kenneth Appel, Wolfgang Haken, and John Koch in 1977. Some history and concern of this proof is given in Section 11.1. Colourings of Planar Maps. In this section, we outline some of the techniques used in the 1977 proof. We also discuss the error made by Alfred Kempe in his 1879 paper.

Note 15.2.A. Appel, Haken, and Koch's proof is by contradiction. Assuming the Four-Colour Theorem is false, there is a smallest (loopless) plane graph which is not 4-colourable. In this section, we denote such a smallest graph as G so that:

(i) G is not 4-colourable, and

(ii) subject to (i), $v(G) + e(G)$ is as small as possible.

Proposition 15.2. Let G be a smallest counterexample to the Four-Colour Theorem. Then

(i) G is 5-critical,

(ii) G is a triangulation, and

(iii) G has no vertex of degree less than four.

Note 15.2.B. In Corollary 10.22 we saw that a simple planar graph has some vertex of degree at most five. If it could be shown that our smallest counterexample G has no such vertex, then we would have a contradiction and the Four-Colour Theorem would be proved. Alfred B. Kempe in his "On the Geographical Problem of the Four Colours," *American Journal of Mathematics*, **2**, 193-200 (1879), reprinted in N. L. Biggs, E. K. Lloyd, and R. J. Wilson's *Graph Theory: 1736–1936*, Oxford University Press (1976), took a step in that direction by extending Proposition 15.2(iii) to show that G has no vertex of degree four.

Theorem 15.3. A smallest counterexample G to the Four-Colour Theorem has no vertex of degree four.

Definition. The paths P_{ij} consisting of vertices of only colours i and j as given in the proof of Theorem 15.3 are *Kempe chains.* The procedure of switching two colours on a Kempe chain is called a *Kempe interchange.*

Note. Kempe chains and Kempe interchange are used to prove the next theorem and its corollary (in Exercise 15.2.1).

Theorem 15.4. A smallest counterexample G to the Four-Colour Theorem contains no separating 4-cycle.

Corollary 15.5. G is 5-connected.

Note. By Corollary 10.22 and Theorem 15.3, smallest counterexample G has a vertex of degree five. Its neighbors induce a 5-cycle (since G is a triangulation by Proposition 15.2(ii)). This cycle is a separating 5-cycle since its removal from G results in separating the degree five vertex from those outside the 5-cycle.

Note. George Birkhoff in "The Reducibility of Maps," *American Journal of Mathematics*, **35**, 115–128 (1913) (available online from JSTOR; accessed 7/22/2022) proved that every separating 5-cycle in a smallest counterexample to the Four-Colour Theorem is induced by the neighbors of a degree five vertex (we accept this without proof). This result, combined with Proposition 15.2(ii) and Corollary 15.5 gives the next theorem. But first, we give a name to the property Birkhoff considered.

Definition. A 5-connected graph with the property that every separating 5-cycle is induced by the neighbors of a degree five vertex is *essentially 6-connected*.

Theorem 15.6. A smallest counterexample G to the Four-Colour Theorem is an essentially 6-connected triangulation.

Note. We now consider Kempe's erroneous proof of the Four-Colour Theorem. This appeared in his 1879 paper mentioned above. In Robin Wilson's popular-level book *Four Colors Suffice: How the Map Problem was Solved* (Princeton University Press, 2002), it is stated that Kempe's mistake is "the most fallacious proof in the whole of mathematics" (see Wilson's Chapter 5). Kempe claimed to have proved that a smallest counterexample cannot contain a vertex of degree five, in contradiction to Corollary 10.22 as mentioned above in Note 15.2.B. Similar to the proof of that G has no vertex of degree four (Theorem 15.3), we assume that v is a vertex of degree five with $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$. Since G is a smallest counterexample then $G - v$ has a 4-colouring (V_1, V_2, V_3, V_4). The plan is to modify this colouring in such a way as to assign at most three colours to the neighbors of v. In this way the fourth colour can be assigned to v yielding a 4-colouring of G and a contradiction.

Note. Consider a 4-colouring of $G - v$. As in the proof of Theorem 15.3, v is adjacent to a vertex of each of the four colours (or else we could assign the fourth colour to v, giving a 4-colouring of G). Without loss of generality, say $v_i \in V_i$ for $1 \leq i \leq 4$, and $v_5 \in V_2$, as in Figure 15.6.

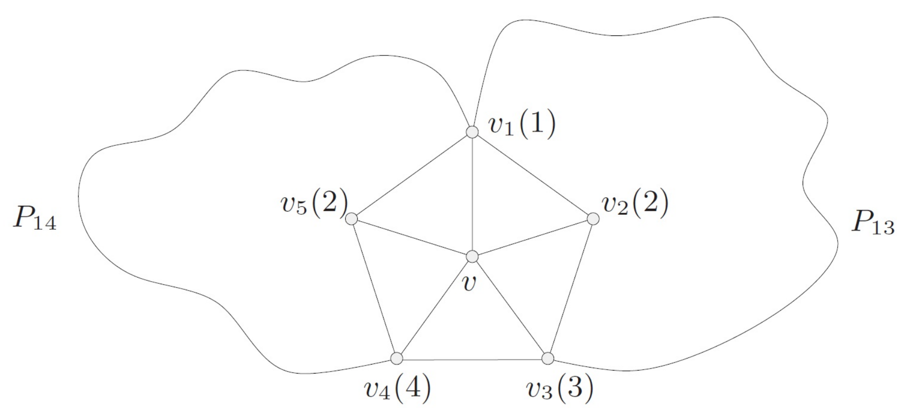
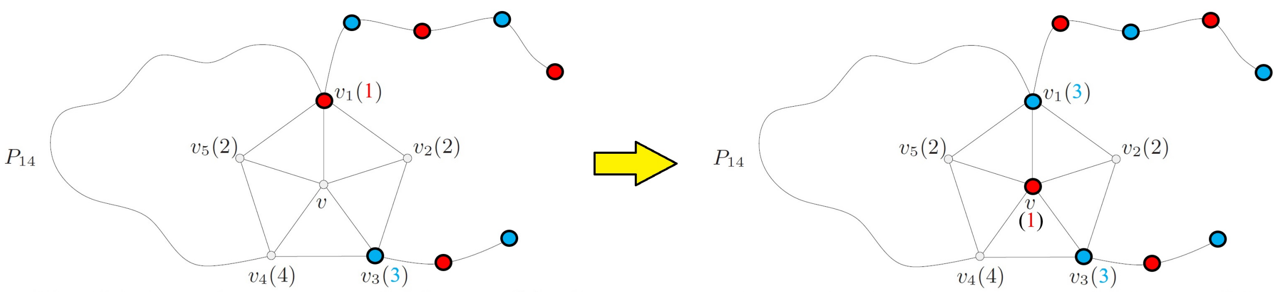


Fig. 15.6. Kempe's erroneous proof of the case $d(v) = 5$

Denote by G_{ij} the subgraph of G induced by the set of vertices $V_i \cup V_j$ (so every vertex of G_{ij} is either colour i or colour j). We may assume that v_1 and v_3 belong to the same connected component of G_{13}, otherwise the colours 1 and 3 on the connected component of G_{13} containing v_1 can be interchanged to give a 4-colouring of $G - v$ where v_1 has colour 3 and so only three colours are assigned to the neighbors of v, implying a 4-colouring of G and a contradiction (see the figure below).

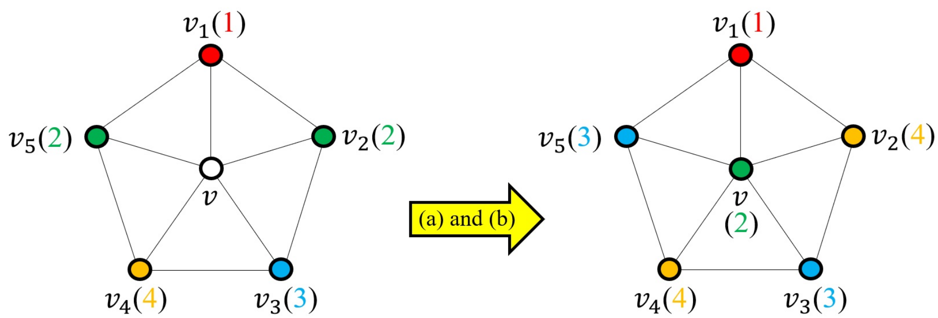


v_1 and v_3 in different connected components of G_{13}

Interchange the colours 1 and 3 on the component of G_{13} containing v_1, then G can be 4-coloured

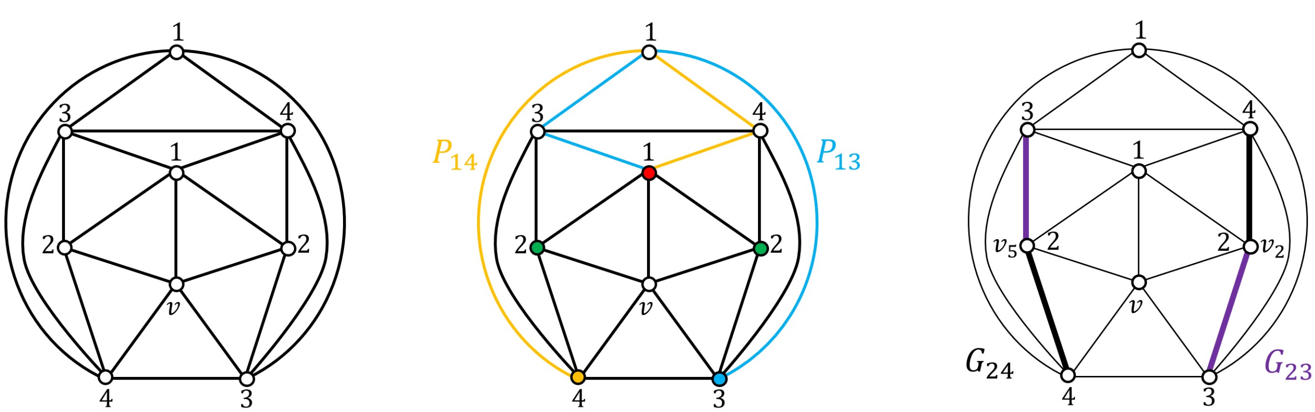
Similarly, we may assume that v_1 and v_4 belong to the same connected component of G_{14}. Let P_{13} be a v_1v_3-path in G_{13} and let P_{14} be a v_1v_4-path in G_{14}. The cycle $vv_1P_{13}v_3v$ separates vertices v_2 and v_4 (in Figure 15.6, $v_2 \in \text{int}(vv_1P_{13}v_3v)$ and $v_4 \in \text{ext}(vv_1P_{13}v_3v)$), and the cycle $vv_1P_{14}v_4v$ separates vertices v_3 and v_5 (in Figure 15.6, $v_5 \in \text{int}(vv_1P_{14}v_4v)$ and $v_3 \in \text{ext}(vv_1P_{14}v_4v)$).

Note. Kempe next argued that the colours 2 and 4 in the component of G_{24} containing v_2 could be interchanged, thus assigning colour 4 to v_2 (since v_2 and v_4 are separated by a cycle, as explained in the previous note, this does not affect the colour of v_4). Similarly, the colours 2 and 3 in the component of G_{23} containing v_5 (v_5 is originally colour 2) could be interchanged, thus assigning colour 3 to v_5 (since v_3 and v_5 are separated by a cycle, this does not affect the colour of v_4). These are the Kempe interchanges. In this way, vertices v_1, v_2, v_3, v_4, v_5 are assigned the colours 1, 3, 4. Then colour 2 can be assigned to vertex v, giving a 4-colouring of G and the desired contradiction. See the figure below.



(a) Interchange the colours 2 and 4 in the G_{24} component containing v_2, and (b) interchange colours 2 and 3 in the G_{23} component containing v_5

As in the proof of Theorem 15.3, one colour interchange causes no problems. But the second colour interchange can "conflict" with the first so that the final vertex colouring is not proper (this can occur when paths P_{13} and P_{14} intersect at internal vertices). This is where the error is. In Exercise 15.2.2, the technique is applied to the following partial colouring of a plane triangulation and results in a vertex colouring that is not proper (notice that P_{13} and P_{14} intersect at the uppermost vertex):



Note. Kempe's error went undetected for about 10 years. Percy J. Heawood in "Map-Colour Theorem," *Quarterly Journal of Pure and Applied Mathematics*, **24**, 332–338 (1890) (this is partially reprinted in N. L. Biggs, E. K. Lloyd, and R. J. Wilson's Graph Theory: 1736–1936, Oxford University Press, 1976) called attention to the fact that the two Kempe interchanges may not result in a proper 4-colouring of $G - v$. Heawood pointed out that the technique could be used to prove the Five-Colour Theorem (with just one Kempe interchange), as we did in the proof of Theorem 11.6.

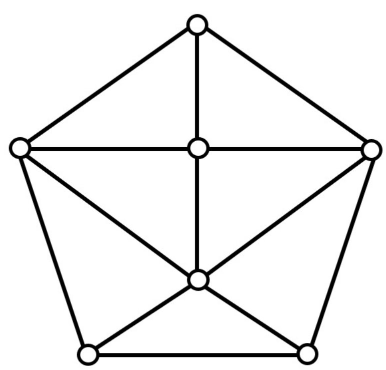
Note. Some of Kempe's ideas were ultimately employed in the 1970s proof of the Four-Colour Theorem. He introduced the ideas of reducibility and unavoidability. We now define these ideas (along with that of a configuration) and illustrate them.

Definition. Let C be a cycle in a simple plane triangulation G. If C has no inner chords and has exactly one inner bridge B, then $B \cup C$ is a *configuration of* G. The cycle C is the *bounding cycle* of the configuration of G and B is its *bridge*. A *configuration* (in general) is a configuration of some simple plane triangulation.

Note. Recall from Exercise 2.2.19 that a chord of a cycle is an edge of a graph not in the cycle which has both of its ends in the cycle. Figure 10.15 of Section 10.4. Bridges illustrates several bridges of a cycle. The wheel W_k with k spokes ($k \geq 2$) is a configuration; the bridge is the center vertex and the spokes (so it's the star S_k) and the vertices of attachment of this bridge make up the set of vertices of the cycle. The Birkhoff diamond of Figure 15.7 below is also a configuration, bounded by a 6-cycle.

Definition. A configuration is *reducible* if it cannot be a configuration of a smallest counterexample to the Four-Colour Conjecture.

Note. Combining Proposition 15.2(iii) and Theorem 15.3, we see that a smallest counterexample to the Four-Colour Theorem G cannot have a vertex of degree four or less. So configurations W_2, W_3, and W_4 are reducible (because the center of W_k is degree k in G, the vertices on the cycle may have higher degree than 4 in G). Theorem 15.6 implies that every separation 5-cycle of G is induced by a degree five vertex. So a separating 5-cycle that is not induced by the neighbors of a degree five vertex, such as the following, is reducible.



Notice that this leaves open as to whether W_5 is a reducible configuration or not. **"Kempe's failed proof was an attempt to show that W_5, also, is reducible"** (Bondy and Murty, page 405). The "Birkhoff diamond" of Figure 15.7 is a reducible configuration as we show in the next theorem. Notice that the 6-cycle on vertices v_i, $1 \leq i \leq 6$, has no chord and has a bridge with four internal vertices and vertices of attachment of v_i for $1 \le i \le 6$.

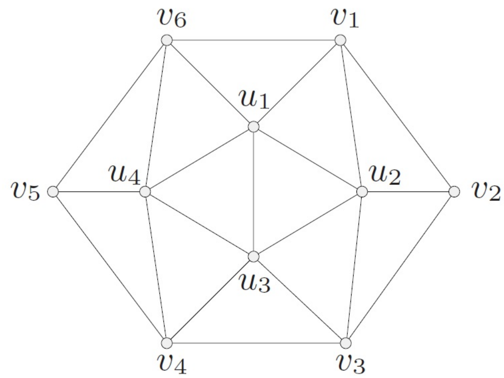


Fig. 15.7. The Birkhoff diamond

Theorem 15.7. The Birkhoff diamond is reducible.

Definition. A set $\mathcal{U}$ of configuration is *unavoidable* if every essentially 6-connected triangulation necessarily contains at least one member of $\mathcal{U}$.

Note. Since (by definition) an essentially 6-connected graph is 5-connected and every separating 5-cycle is induced by the neighbors of a degree five vertex, then the set $\{W_5\}$ is an unavoidable set. By the definition of reducible configuration, a smallest counterexample to the Four-Colour Theorem cannot contain a reducible configuration. Since a smallest counterexample is essentially 6-connected by Theorem 15.6 and is a triangulation by Theorem 15.2(ii), then such a counterexample must contain at least one configuration from each unavoidable set. HERE'S THE PUNCHLINE: To prove that no counterexample exists, it suffices to find an unavoidable set of configurations, each of which is reducible! We would then have a counterexample that must contain a reducible configuration (which, of course, it cannot contain). **This** contradiction proves that no counterexample can exist, and hence the Four-Colour Theorem holds. The unavoidable set of configurations given by K. Appel and W. Haken in "Every Planar Map is Four Colorable. I. Discharging," *Illinois Journal of Mathematics*, **21**, 429–490 (1977) (available online on the Project Euclid webpage; accessed 7/25/2022), had 1482 members. N. Robertson, D. Sanders, P. Seymour, and R. Thomas, "The Four-Colour Theorem," *Journal of Combinatorial Theory, series B*, **70**, 166–183 (1997) (available online on ScienceDirect.com; accessed 12/31/2022), using a technique more refined than that of Appel and Haken, constructed an unavoidable set with only 633 members.

Note. Finding unavoidable sets is addressed by the process of "discharging." Each vertex is assigned a "charge," and this charge is redistributed by a "discharging algorithm" that defines a set $\mathcal{U}$ of configurations such that any triangulation which contains no member of $\mathcal{U}$ is discharged by the algorithm.

Definition. Given a graph G, for each vertex $v \in V(G)$ the weight $6 - d(v)$ is assigned to v is the *charge* of v (the charge need not be an integer). To *discharge* a vertex of graph G is to attempt to redistribute the charge in some "methodical way" (that is, discharging algorithm) that makes the charge at the vertex negative or zero.

Note. We will not explore the details on how we know that the discharging algorithm produces configurations that are unavoidable by triangulations. Instead we illustrate by example how a discharging algorithm modifies charges and discharges vertices.

Note. No discharging algorithm can completely discharge the vertices of a triangulation because the sum of charges (which remains the same in the algorithm; it "conserves" total charge) is always positive:

$$\begin{aligned}\text{total charge} &= \sum_{v\in V}(6-d(v)) = 6v(G) - \sum_{v\in V} d(v)\\ &= 6v(G) - 2e(G) = 6n - 2m\\ &= 6n - 2(2n-6) \text{ since } m = 3n-6 \text{ for a triangulation by}\\ &\quad\ \text{Corollary 10.21; this follows from Euler's Formula (Thm. 10.19)}\\ &= 12.\end{aligned}$$

So any triangulation must contain at least one member of $\mathcal{U}$; that is, $\mathcal{U}$ is an unavoidable set of configurations. We arrive at this conclusion by ASSUMING a triangulation contains *no* members of $\mathcal{U}$ and use a discharging algorithm to discharge all vertices, which is a CONTRADICTION (since then the sum of the charges is nonpositive, but we know that it is in fact 12). Hence the triangulation must contain members of the unavoidable set $\mathcal{U}$.

Note. In the next note, we give an example of a discharging algorithm. A more detailed explanation of the same algorithm is given in my supplement to this section, Supplement. The Four-Color Theorem: A History, Part 2 (see Note FCT.N). This supplement gives a more detailed and visual explanation of what is presented next, and it is recommended that read the supplement before reading what follows. Beware, though, that the supplement considers the map itself and the regions are charged, as opposed to the presentation here where the charge is on the vertices (because we are considering the dual of the map itself).

Note. As an example, consider a discharging algorithm that takes the charge of each vertex of degree five and distributes its charge of one equally amongst its five neighbors. A vertex v of degree eight or more is discharged by this algorithm because the maximum charge that a vertex can receive from its neighbors is $\frac{1}{5}d(v)$ (in the case that each neighbor of v is of degree five), so for $d(v) \geq 8$ we have the new charge of v is at most

$$(6 - d(v)) + \frac{1}{5}d(v) = 6 - \frac{4}{5}d(v) \leq 6 - \frac{4}{5}(8) = -2/5 < 0.$$

A vertex v of degree seven with no more than five neighbors of degree five is discharged because the new charge of v is at most $(6 - d(v)) + \frac{1}{5}(5) = (-1) + (1) = 0$. If v is a vertex of degree five with no neighbor of degree five then the new charge is $(6 - d(v)) = (6 - d(v) = (1) - (1) = 0$ and such a vertex is discharged. A vertex v of degree six with no neighbors of degree five has a charge of $6 - d(v) = 6 - 6 = 0$ which remains unchanged, and such a vertex is discharged. So a triangulation consisting only of vertices satisfying these conditions is discharged by this algorithm. That is, all vertices have a new charge that is either negative or zero. But we know by the previous note that the total charge is 12 and that the discharging algorithm preserves charge, so having all vertices with nonpositive charge leads to a contradiction. In an essentially 6-connected triangulation, we consider the set $\mathcal{U}$ of configurations with:

(1) a vertex of degree five that is adjacent to a vertex of degree five,

(2) a vertex of degree five that is adjacent to a vertex of degree six, and

(3) a vertex of degree seven with at least five neighbors of degree five.

However, in (3) the vertex of degree seven must have two consecutive neighbors of degree five and these are adjacent in G. Therefore case (3) is included in case (1). So the set $\mathcal{U}$ of unavoidable configurations for this discharging algorithm includes the following two configurations (1) and (2), which we represent as:

Here we have used the "Heesch representation" of the configuration in which the numbers indicate degrees.

Note. The discharging technique has been applied to other colouring problems of planar graphs and graphs embeddable on other surfaces. R. Steinburn conjectured that every planar graph without cycles of length four or five is 3-colourable. H.L. Abbott and B. Zhou in "On Small Faces in 4-Critical Planar Graphs," *Ars Combinatoria*, **32**, 203–207 (1991) proved a weaker version of Steinburg's Conjecture by considering graphs with no cycles of length k for all $4 \leq k \leq 11$.

Theorem 15.2.A. A planar graph is 3-colourable if it contains no cycles of length k for $4 \leq k \leq 11$.

Note. O.V. Borodin, A.N. Glebov, A. Respaud, and M.R. Salavatipour in "Planar Graphs without Cycles of Length 4 to 7 are 3-Colourable," *Journal of Combinatorial Theory, Series B*, **93**, 303–311 (2005) (available on the ScienceDirect webpage, accessed 7/27/2022) proved that Theorem 15.2.A could be refined by requiring

$4 \leq k \leq 7$. They also used discharging but with more complicated discharging algorithms. Y. Zhao in "3-Coloring Graphs Embedded in Surfaces," *Journal of Graph Theory*, **33**, 140–143 (2000) (some details, and possible access, are online on the Journal of Graph Theory webpage, accessed 7/27/2022) extended Theorem 15.2.A to closed surface Σ by proving there exists some constant $f(\Sigma)$ (dependent only on surface Σ) such that any graph embeddable on Σ and containing no k-cycles for $4 \leq k \leq f(\Sigma)$, is 3-colourable.

Note. In the two 1977 papers of Appel, Hakin, and Koch presenting a proof of the Four-Colour Theorem, 487 discharging rules were used which resulted in over 1400 unavoidable configurations; a computer search generated the 1400-odd configurations. N. Robertson, D. Sanders, P. Seymour, and R. Thomas in "The Four-Colour Theorem," *Journal of Combinatorial Theory, Series B*, **70**, 2–44 (1997) using "only" 32 discharging rules to find 633 unavoidable configurations (again, with computer assistance; available online on the Science Direct website, accessed 1/11/2023).

Note. The Appel, Hakin, and Koch 1977 proof drew quick criticism. Since key parts of the proof were dependent on a computer search that could not be checked by hand, the proof itself could not be checked. In the first of the two 1977, K. Appel and W. Haken, "Every Planar Map is Four Colorable. I. Discharging," *Illinois Journal of Mathematics*, **21**, 429-490 (1977), the discharging methods for constructing the unavoidable sets are described. In the second paper, K. Appel, W. Haken, and

J. Koch, "Every Planar Map is Four Colorable. Part II. Reducibility," *Illinois Journal of Mathematics*, **21**, 491-567 (1977), the computer program is described and the entire unavoidable set of reducible configurations are listed. The two papers were supplemented by a microfiche of 450 additional pages of additional diagrams and explanations. The controversy receded some with the publication of Appel and Haken's *Every Planar Map is Four Colorable*, Contemporary Mathematics #98, 741 pp., American Mathematical Society (1989). This work is described on the AMS Bookstore webpage as:

> "...the book contains the full text of the supplements and checklists, which originally appeared on microfiche. The thirty-page introduction, intended for nonspecialists, provides some historical background of the theorem and details of the authors' proof. In addition, the authors have added an appendix which treats in much greater detail the argument for situations in which reducible configurations are immersed rather than embedded in triangulations. This result leads to a proof that four coloring can be accomplished in polynomial time." (Accessed 9/4/2022.)

The relatively simple 1997 *JCT-B* paper of N. Robertson, D. Sanders, P. Seymour, and R. Thomas also calmed things. In 2008 (the same as the publication of Bondy and Murty's graduate text book), George Gonthier published "Formal Proof—The Four Color Theorem," *Notices of the American Mathematical Society*, **55**(1), 1382–1393 (accessed 9/4/2022). In this work, Gonthier (and colleague Benjamin Werner) formalized a proof in the "COQ" (for "Calculus of Constructions," with a little French to change the second C into a Q) scheme. The Wikipedia COQ page (accessed 9/4/2022) describes COQ as:

> "Coq is an interactive theorem prover first released in 1989. It allows for expressing mathematical assertions, mechanically checks proofs of these assertions, helps find formal proofs, and extracts a certified program from the constructive proof of its formal specification. Coq works within the theory of the calculus of inductive constructions, a derivative of the calculus of constructions. Coq is not an automated theorem prover but includes automatic theorem proving tactics (procedures) and various decision procedures."

Gonthier's paper removes the need to trust the computer program, but it shifts this over to a requirement to trust the COQ scheme.

Note. The complaints that computer techniques have intruded into pure mathematics proofs still persist. One resolution for the Four Color Theorem would be to introduce a non-computer-based proof. But this would require new ideas and, over the past 45 to 50 years since Appel and Haken's initial success, there seems to be no progress in this direction. The mathematics writer Ian Stewart is paraphrased in Wilson's *Four Colors Suffice* (2002) as complaining that the Appel-Haken proof "did not explain *why* the theorem is true—partly because it was too long for anyone to grasp all the details, and partly because it seemed to have no structure" (see page 220). The purpose of a mathematical proof is not so much to learn *what* is true about mathematical structures, but *why* the mathematical structures have the proven properties.

Revised: 4/20/2023