Section 15.3. List Colouring of Planar Graphs

Note. In this section, we consider list colourings for planar graphs which are "near-triangulations" of the plane. Our main result will give an alternate proof of the Five Colour Theorem (Theorem 11.6). We'll discuss attempts to use list colourings to prove the Four Colour Theorem (thus potentially giving a concise, readable proof of it).

Definition. A *near-triangulation* is a plane graph all of whose inner faces have degree three.

Note. The next result is due to Carsten Thomassen in "Every Planar Graph is 5-Choosable," *Journal of Combinatorial Theory, Series B*, **62**, 180-181 (1994). Thomassen is the coauthor (along with Gojan Mohar) of *Graphs on Surfaces* (Johns Hopkins University Press, 2001), the source for my in-preparation online notes for Topological Graph Theory (not a formal ETSU class).

Theorem 15.3.8. Let G be a near-triangulation whose outer face is bounded by a cycle of C, and let x and y be consecutive vertices of C. Suppose that $L : V \to 2^{\mathbb{N}}$ (where $2^{\mathbb{N}}$denotes the power set of $\mathbb{N}$) is an assignment of lists of colours to the vertices of G such that:

(i) $|L(x)| = |L(y)| = 1$, where $L(x) \neq L(y)$,

(ii) $|L(v)| \geq 3$ for all $v \in V(C) \setminus \{x, y\}$, and

(iii) $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$.

Then G is L-colourable.

Note. Recall that every simple graph on three or more vertices is a spanning graph of a triangulation by Exercise 10.2.3). So any (simple) planar graph is a spanning graph of a triangulation and if we use define list L where $L(x) = \{1\}$, $L(y) = \{2\}$, (where x and y are consecutive on cycle C of Theorem 15.8), and $L(v) = \{1, 2, 3, 4, 5\}$ for all $v \in V(G) \setminus \{x, y\}$. Then Theorem 15.8 implies that the triangulation is 5-list colourable and hence the original planar graph is 5-list colourable and therefore is 5-colourable. This gives us a proof of the following and another proof of the Five Colour Theorem (Theorem 11.6).

Corollary 15.9. Every planar graph is 5-list colourable.

Note. It is reasonable to wonder if one could mimic the proof of Theorem 15.8 to get a condition that would allow us to show that every planar graph is 4-list colourable. This would then give us a clean, readable proof of the Four Colour Theorem. However, M. Voigt showed in "List Colourings of Planar Graphs," *Discrete Mathematics*, **120**, 215–219 (1993) that there are planar graphs which are not 4-list colourable. Voigt's paper is online on the ScienceDirect website (accessed 7/18/2022). Bondy and Murty comment (see page 412): "Even so, it is conceivable that an appropriate list colouring version of the Four Colour Theorem will provide a more transparent (and shorter) proof of that theorem, as well."

Theorem 15.10. Grötzch's Theorem.

Every triangle-free graph is 3-colourable.

Note. In the spirit of this section and Corollary 15.9, we might wonder if Grötzch's Theorem holds if we replace "3-colourable" with "3-list colourable." In fact, Voigt (in 1995 in a different paper from that mentioned above) gave examples of triangle-free planar graphs that are not 3-list colourable. Now Thomassen (in the same paper in which he proved Theorem 15.8) proved that every planar graph of girth five is 3-list-colourable. Grötzch's Theorem can be reduced from a result on triangle-free planar graphs to a result on girth four planar graphs (see the statement of Grötzch's Theorem in Mohar and Thomassen's *Graphs on Surfaces* as Theorem 8.5.1). With this interpretation, Thomassen's result can be viewed as the list-colouring extension of Grötzch's Theorem.

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