Section 15.4. Hadwiger's Conjecture

Note. In this section, we concentrate on minors of k-chromatic graphs. This section appears to more appropriately belong to the topics of Chapter 14 (Vertex Colourings). The conversation in this section will include mention of the Four Colour Theorem and the concept of a graph minor first appeared in out classification of planar graphs in Section 10.5. Kuratowski's Theorem; otherwise, this section is independent of the other vertices in this chapter.

Note. Recall that a *minor* of a graph G is any graph obtainable from G through a sequence of vertex and edge deletions and edge contractions.

Note. We saw in Note 14.1.B that if graph G contains a clique on r vertices (i.e., if K_r is a subgraph of G), then $\chi(G) \geq r$. However, a graph may have chromatic number k and yet not have a K_k subgraph. Consider the 4-chromatic Hojós graph of Figure 14.1, for example.

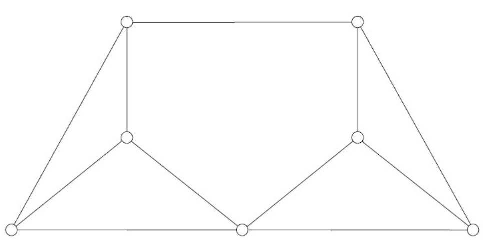

Fig. 14.1. The Hajós graph: a 4-chromatic graph

Hugo Hadwiger in "Über eine Klassifikation der Streckenkomplexe [On a Classification of the Route Complexes]," *Vierteljschr. Naturforschende Gesellschaft in Zürich*, **88**, 133-142 (1943) (a copy is online on the Naturforschende Gesellschaft in

Zürich webpage; accessed 7/20/2022) conjectures that a k-chromatic graph must contain a clique on k vertices, but not as a subgraph but instead as a minor.

Conjecture 15.11. Hadwiger's Conjecture.
Every k-chromatic graph has a K_k-minor.

Note. For $k = 1$ and $k = 2$, Hadwiger's Conjecture clearly holds (since K_1 is a subgraph of every graph and K_2 is a subgraph of every nonempty graph). For $k = 3$, a 3-chromatic graph necessarily contains an odd cycle (this follows from the fact that a 3-chromatic graph has some 3-critical subgraph and by Exercise 14.2.2, 3-critical graphs are cycles of odd length) and an odd cycle contains K_3 as a minor so that the conjecture holds for $k = 3$. Hadwiger proved the case $k = 4$ in his 1943 paper. G.A. Dirac proved the following stronger version of Hadwiger's Conjecture is the case $k = 4$ (in 1952).

Theorem 15.12. Every 4-chromatic graph contains a K_4-subdivision.

Note. In 1964 K. Wagner in "Beweis einer Abschwäching der Hadwiger-Vermutung [Proof of a Weakening of Hadwiger's Conjecture]," *Mathematische Annalen*, **153**, 139-141 (1964) (a copy can be accessed online through the European Digital Mathematics Library; accessed 7/20/2022) proved that for $k = 5$, Hadwiger's Conjecture is equivalent to the Four Colour Theorem (and hence holds, given Appel, Haken, and Kock's proof of the Four Colour Theorem in 1977). The case $k = 6$ was proved (also using the Four Colour Thoerem by N. Robertson, P. Seymour, and R. Thomas in "Hadwiger's Conjecture for K_6-free Graphs," *Combinatorica*, **13**, 279–361 (1993). The conjecture remains unsolved 80 years after it was first stated. According to the Wikipedia page on Hadwiger's Conjecture (accessed 7/20/2022), it "is considered to be one of the most important and challenging open problems in the field [of graph theory]."

Note. W. Mader in "Homomorphieeigenschafter und mittlere Kantendichte von Graphen [Homomorphism Property and Mean Edge Density of Graphs]," *Mathematische Annalen*, **174**, 265–268 (1967) proved the following weaker form of Hadwiger's Conjecture. It shows that every graph with sufficiently many edges (and hence with sufficiently high chromatic number) has a K_k-minor.

Theorem 15.13. Every simple graph G with $m \geq 2^{k-3}n$ (that is, $e(G) \geq 2^{k-2}v(G)$) has a K_k-minor.

Corollary 15.14. For $k \geq 2$, every $(2^{k-2} + 1)$-chromatic graph has a K_k-minor.

Note. We now discuss a very unsuccessful conjecture. In the early 1950s, G. Hajós conjectured that every k-chromatic graph G contains a subdivision of K_k (the exact origins of the conjecture are murky). Since a subdivision of K_k can, through a sequence of edge contractions and edge deletions, be used to produce a K_k-minor of the subdivision of K_k (and hence a minor of G itself), then Hajós Conjecture implies

Hadwiger's Conjecture. Also, the case $k = 4$ of Hajós' Conjecture is given by Theorem 15.12. But Paul Catlin in "Hajós' Graph-Coloring Conjecture: Variations and Counterexamples," *Journal of Combinatorial Theory, Series B*, **26**, 268–274 (1979) (available online on the ScienceDirect webpage, accessed 7/21/2022) gave an 8-chromatic graph which contains no subdivision of K_8 (see Figure 14.3 and Exercise 15.4.3). Hence, the Hajós Conjecture is false. Worse yet, Paul Erdős and Siemion Fajtlowicz in "On the Conjecture of Hajós," *Combinatorica*, **1**(2), 141–143 (1981) (available online on the CiteSeerX website, accessed 7/21/2022) using the probabilistic method proved that "almost every graph" is a counterexample to the Hajós Conjecture. We now formally state this result. We omit the proof; there is a limit claim in Bondy and Murty's proof that is (in the humble opinion of your instructor) suspicious...

Theorem 15.15. Almost every graph is a counterexample to Hajós Conjecture.

Note. It seems that a number of counterexamples to Hajós Conjecture are known, and some of them even predate the the first known statement of the conjecture. This is discussed in Carsten Thomassen's "Some Remarks on Hajós' Conjecture," *Journal of Combinatorial Theory, Series B*, **93**, 95–105 (2005) (available online on the ScienceDirect webpage, accessed 7/21/2022). But this was not widely known until relatively recently. In Bondy and Murty's *Graph Theory with Applications*, NY: North-Holland (1976), section 8.3 is on the Hajós' Conjecture. In this section, they state (on page 124): "Hajós conjecture has not yet been settled in general, and its resolution is known to be a very difficult problem. There is a related conjecture due to Hadwiger (1943): if G is k-chromatic, then G is 'contractible' to a graph which contains K_k." We see that graph theory is a fast-growing and evolving area! In the time between 1976 and 2008, Bondy and Murty shifted their focus from the (now known to be false) Hajós Conjecture to the (still open) Hadwiger Conjecture.

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