Chapter 16. Matchings

Section 16.1. Maximum Matchings

Note. In this section we considering "pairing up" vertices of a graph using a subset of the edges. We define types of such pairings, state an application, and consider the problem of finding one with the most pairs.

Definition. A *matching* in a graph is a set of pairwise nonadjacent links (i.e., nonloop edges). If M is a matching, the two ends of each edge of M are said to be *matched* under M and each vertex incident with an edge of M is *covered* by M. A *perfect matching* is one which covers every vertex of the graph. A *maximum packing* is one which covers as many vertices (in a given graph) as possible. A graph is *matchable* if it has a perfect matching.

Note. The number of edges in a maximum matching of graph G is the *matching number* of G, denoted $\alpha'(G)$. A *maximal matching* is one which cannot be extended to a larger matching.

Note. Of course, maximal matchings may not be maximum matchings. In Figure 16.1 (below), a maximal matching of four edges of the pentagonal prism is given (left) and a perfect matching of five edges of the same graph is also given (right).

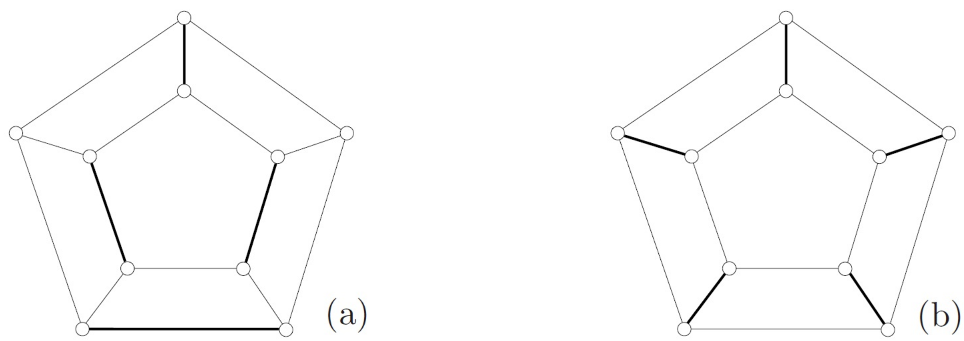

Fig. 16.1. (a) A maximal matching, (b) a perfect matching

Note. We address the problem of finding maximum matchings of a graph. Formally, we consider the following.

Problem 16.1. The Maximum Matching Problem.

GIVEN: A graph G.

FIND: A maximum matching M^* in G.

Note. An application of the Maximum Matching Problem is the assigning of objects in one set to another set of categories where some requirement is put on the objects in terms of the categories to which they can be assigned. An example is the following.

Problem 16.2. The Assignment Problem.

A certain number of jobs are available to be filled. Given a group of applicants for there jobs, fill as many of them as possible, assigning applicants only to jobs for which they are qualified.

Note. Problem 16.2 is called the Scheduling Problem in my online notes for Introduction to Graph Theory (MATH 4347/5347); see Section 7.2, "Matchings in Graphs, Scheduling Problems." The problem is addressed by introducing a bipartite graph $G[X, Y]$ where X represents the applicants and Y represents the jobs. As edge xy is in the graph of applicant x is qualified for job y. A maximum matching in G gives a way to fill as many of the jobs by qualified applicants as possible. In Section 16.5, "Matching Algorithms," we will see a polynomial time algorithm, Egarváry's Algorithm, that solves the Assignment Problem; that is, it gives a maximum matching in a bipartite graph. In fact, we will see a polynomial time algorithm, Edmond's Algorithm, that gives a maximum matching in an arbitrary graph. The following definition involves ideas that play a role in these algorithms.

Definition. Let M be a matching in a graph G. An *M-alternating path* or *M-alternating cycle* in G is a path or cycle where edges are alternately in M and $E \setminus M$. An M-alternating path in which neither its origin nor its terminus is covered by M is an *M-augmenting path.*

Note. Figure 16.2 shows some M-alternating paths, including an M-augmented path (uppser left).

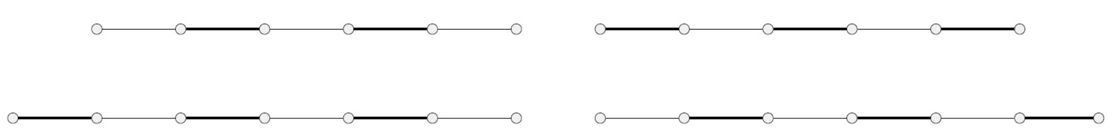

Fig. 16.2. Types of M-alternating paths

Figure 16.3(a) shows an M-augmented path along with the maximal matching (which is not a maximum matching) from Figure 16.1. The existence of this M-augmented path illustrates the next theorem, which relates augmented paths to maximum matchings.

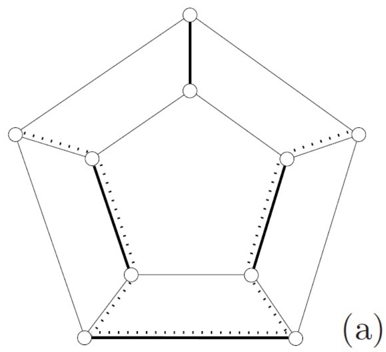

Fig. 16.3. (a) An M-augmenting path P

Theorem 16.3. Berge's Theorem.

A matching M in a graph G is a maximum matching if and only if G contains no M-augmenting path.

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